Problem 1. Let $k \geqslant 2$ be an integer. Let m and n be coprime positive integers with exactly k positive divisors such that $m < n$.

For $i \in \{1, \ldots, k\}$, denote by f_i and d_i the i-th smallest divisor of m and n, respectively. Suppose that

$$d_i - f_i \mid n - m$$

for all $i \in \{2, \ldots, k\}$. Prove that $d_i \geqslant f_i$ for all $i \leqslant \frac{k}{2}$.

(Ivan Novak)

First Solution. We have by the condition of the problem:

$$f_1 < f_2 < \cdots < f_k \quad and \quad d_1 < d_2 < \cdots < d_k$$

Pairing up the divisors, we get:

$$n = d_{k+1-j} d_j \quad and \quad m = f_{k+1-j} f_j, \quad \forall j \in \{1, 2, \cdots, k\}$$

1 point.

Suppose for the sake of contradiction that there is $i \leq \frac{k}{2}$ such that $d_i < f_i$. So, we have:

$$j = k + 1 - i \geq k + 1 - \frac{k}{2} \geq \frac{k}{2} + 1 \geq 2$$

The last inequality implies $f_i < f_j$ and $d_j > f_j$, otherwise $m > n$ which is impossible.

1 point.

Setting $a = f_i - d_i$ and $b = d_j - f_j$, we see that both a and b are positive integers and by the condition of the problem we have:

$$a \mid n - m \quad and \quad b \mid n - m$$

We have now: $n - m = d_i d_j - f_i f_j = (d_i - f_i) d_j + (d_j - f_j) f_i = -a d_j + b f_i$. The last equation gives:

$$a \mid f_i b \quad and \quad b \mid d_j a$$

3 points.

But since $gcd(m, n) = 1$, then we get:

$$1 = gcd(f_t, d_j) = gcd(f_i, f_i - d_j) = gcd(f_i, a)$$

1 point.

We deduce that a divides b and by symmetry that b divides a, hence $a = b$.

1 point.

Returning to the initial equation and plugging this result gives:

$$n - m = a(f_i - d_j) = a(f_i - f_j - a)$$

But $a(f_i - f_j - a) < 0$ because $f_i - f_j \leq 0$ and $a > 0$, so $n - m < 0$ which is a contradiction, thus such i does not exist which solves the problem. ∎

3 points.

Second Solution. For the sake of contradiction, suppose $f_i > d_i$ for some $i \leq \frac{k}{2}$. Then, since

$$n = d_i \cdot d_{k+1-i} > f_i \cdot f_{k+1-i} = m$$

1 point.

we must have $d_{k+1-i} > f_{k+1-i}$.

1 point.

We notice that:

$$d_i - f_i \mid n - m$$
$$\implies d_i - f_i \mid n - m - \frac{n}{d_i} \cdot (d_i - f_i)$$
$$\implies d_i - f_i \mid \frac{n \cdot f_i}{d_i} - \frac{m \cdot f_i}{f_i}$$
$$\implies d_i - f_i \mid \frac{n}{d_i} - \frac{m}{f_i}$$
$$\implies d_i - f_i \mid d_{k+1-i} - f_{k+1-i}$$

4 points.

This gives us
$$d_i - f_i \mid d_{k+1-i} - f_{k+1-i} \quad \text{and} \quad d_{k+1-i} - f_{d+k-1} \mid d_i - f_i \ \text{(in a similar way)}$$

1 point.

From which we deduce that
$$f_i - d_i = d_{k+1-i} - f_{k+1-i} \iff \frac{n}{d_i} + d_i = \frac{m}{f_i} + f_i$$

Now since $n > m$ we have:
$$\frac{n}{d_i} + d_i > \frac{m}{d_i} + d_i$$

And from $(a+b)^2 = (a-b)^2 + 4ab$ and noting that since we supposed $d_i < f_i$ we have $\frac{m}{d_i} - d_i > \frac{m}{f_i} - f_i$ we have:
$$\frac{m}{f_i} + f_i < \frac{m}{d_i} + d_i$$

The last two inequalities give us:
$$\frac{m}{d_i} + d_i < \frac{n}{d_i} + d_i = \frac{m}{f_i} + f_i < \frac{m}{d_i} + d_i$$

which is a contradiction. ■

3 points.

Problem 2. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function such that for every positive integer k, the set $\{f(1), f(2), \ldots, f(k)\}$ contains exactly $f(f(k))$ elements. Prove that

$$f(f(f(k))) = f(k)$$

for every positive integer k.

(Ivan Novak)

First Solution. Note that $f \circ f$ is increasing and $f(f(n+1)) - f(f(n)) \leqslant 1$ for all $n \geqslant 1$.

1 point.

Therefore, the image of $f \circ f$ is either the whole of $\mathbb{N}$, or a set of the form $\{1, 2, \ldots, a\}$ for some $a \geqslant 1$.

If the image is the whole of $\mathbb{N}$, let $a_1, a_2, \ldots$ be a sequence such that a_i is the smallest positive integer satisfying $f(f(a_i)) = i$. Note that the image of f is also $\mathbb{N}$. Also, if $f(x)$ is distinct from $f(1), f(2), \ldots, f(x-1)$, then $f(f(x)) > f(f(x-1))$. This means that x is an element of the sequence $a_1, a_2, \ldots$. From this we can conclude that the image of f is equal to $\{f(a_1), f(a_2), \ldots, \}$.

Now for any $x \in \mathbb{N}$, if $f(a_j) = x$ for some j, we must have $f(x) = j$. Hence, if $f(x) = f(y) = j$ for some positive integers j, y and x, we must also have $f(a_j) = x = y$. Therefore, f is injective. Now it's easy to see that $f(f(n)) = n$, from which the claim follows.

4 points.

Suppose the image of $f \circ f$ is equal to $\{1, \ldots, a\}$ for some $a \geqslant 1$. Then if the image of f contained more than a elements, taking the least n such that $\{f(1), \ldots, f(n)\}$ contains $a + 1$ elements would yield a contradiction. Hence, the image of f is equal to $\{1, \ldots, a\}$ as well. However, this means that

$$\{1, \ldots, a\} = f(\mathbb{N}) = f(f(\mathbb{N})) = f(\{1, \ldots, a\}),$$

which means that the restriction of f to the set $\{1, \ldots, a\}$ is a bijection. Now it's easy to see from the condition that $f(f(j)) = j$ for all $j \leqslant a$. For any $j > a$, we must have $f(f(j)) = a$ and $f(j) \leqslant a$. Hence, $f(j)$ is the unique integer $k \leqslant a$ such that $f(k) = a$. Then we have

$$f(f(f(j))) = f(a) = f(f(k)) = k = f(j),$$

which proves the claim.

5 points.

Second Solution. Like in previous solution we notice that $0 \leq f(f(n+1)) - f(f(n)) \leq 1$.

1 point.

Lemma 1: $f(f(a)) = f(f(b)) \implies f(a) = f(b)$
Let k and l respectively be the smallest integers such that $f(k) = f(a)$ and $f(l) = f(b)$. Then we have

$$f(f(k)) = f(f(a)) = f(f(b)) = f(f(l)).$$

1 point.

Now, if $k > l$, we have $f(f(k)) > f(f(l))$ because $f(k) \notin \{f(1), f(2), \ldots, f(k-1)\}$ (we know that $k-1$ exists since $k > l$), so it must be that $f(f(k)) > f(f(l))$, and $f \circ f$ is increasing. Similarly, we solve $l > k$. So we must have $k = l$ which implies

$$f(a) = f(k) = f(l) = f(b).$$

5 points.

We do a proof by induction.

Base $k = 1$: It is obvious that $f(f(1)) = 1$ from problem statement. So we have $f(f(f(1))) = f(1)$.

Assumption for all $k \leq n$: Assume that $f(f(f(k))) = f(k)$.

Step for $k = n+1$: We will split this in 2 cases.

Case 1: $f(f(n+1)) = f(f(n))$.
Using the lemma, we have $f(n+1) = f(n)$. And then we have $f(f(f(n+1))) = \underbrace{f(f(f(n))) = f(n)}_{Assumption} = f(n+1)$.

1 point.

Case 2: $f(f(n+1)) = f(f(n)) + 1$.
If $f(f(n)) = n$, this is trivial since $f(f(f(n+1))) = f(f(f(n)) + 1) = f(n+1)$.
Assume that $f(f(n)) < n$. This implies $f(f(n+1)) \leq n$ so we can use the assumption of induction for selecting $k = f(f(n+1))$, so we get $f(f(f(f(f(n+1))))) = f(f(f(n+1)))$. Using the lemma twice, we get $f(f(f(n+1))) = f(n+1)$.

2 points.

Third Solution. Let n be the biggest integer such that $f(f(n)) = n$ if it exists. Otherwise since we know $f(f(1)) = 1$ it implies that it must hold for all $n \in \mathbb{N}$ so we have $f(f(f(k))) = f(k)$ for all $k \in \mathbb{N}$.

1 point.

By the fact that n is the biggest such integer, we must have $f(f(n+1)) = f(f(n)) = n$ because, as in previous solutions, we have $0 \leq f(f(n+1)) - f(f(n)) \leq 1$.

1 point.

By the definition of n and the condition from problem we have $f(f(i)) = i$ for all $i \leq n$ this gives $f(f(f(i))) = f(i)$ which means that $f(i)$ is a fixed point of $f \circ f$ and since n is the biggest fixed point of $f \circ f$ by our definition we have $f(i) \leq n$.

1 point.

We will prove that $f(m) = f(n)$ for all $m > n$. Assume that $m > n$ is the first integer such that $f(m) \neq f(n)$ this means that

$$f(n) = f(n+1) = \cdots = f(m-1)$$

Case 1: $f(f(m)) = f(f(n))$.
We can proceed to prove it as in the **Lemma 1 of Second Solution** that we must have $f(m) = f(n)$. However, due to many more assumptions, easier proofs also work.

3 points.

Case 2: $f(f(m)) = f(f(n)) + 1 = n + 1$ Considering $f(f(f(m))) = f(n+1) \leq n < n + 1 = f(f(m))$ we must have $f(m) < m$. But now we get

$$f(f(m)) \in \{f(1), f(2), \ldots, f(m-1)\} = \{f(1), f(2), \ldots, f(n)\}$$

. And since we know that all of those numbers are less than n (as we have proven). We get $n + 1 = f(f(n+m)) < n$, which is a contradiction. ∎

4 points.

Problem 3. Let ABC be an acute triangle with circumcircle ω. Let the angle bisector of $\angle B$ intersect AC, ω, and the parallel to AB from C in D, E and F respectively. Let X be the intersection of ω and the circumcircle of triangle $\triangle DCF$ and let Y be a point on CF such that $YF = YD$. The line XF intersects ω and DY in T and P respectively. The circumcircle of triangle $\triangle TDE$ meets the lines PF and EY in R and S.
Prove that the circumcircles of triangles $\triangle PRS$ and $\triangle BDC$ are internally tangent.

(Yasser Merabet)

Solution. Define the point R as the intersection of the circumcircle of triangle $\triangle TDE$ with the line PF, and the point S as the intersection of the circumcircles of triangles $\triangle TDE$ with the line EY. We will show that the circumcircles of triangles $\triangle PRS$ and $\triangle BDC$ are internally tangent which solves the problem. To do so, define the point Z as the intersection of the lines PB and CF.

Let $\angle$ denote directed angles throughout this solution, (signed mod $180°$). The notation (AB) means the line which passes through the points A and B and the notation (ABC) means the circumcircle of triangle $\triangle ABC$.
Claim 1: $(XB) \parallel (RD)$
Proof: We have:
$$\underbrace{\angle EDR = \angle ETR}_{RTDE \text{ cyclic}} = \underbrace{\angle ETX = \angle EBX}_{XTBE \text{ cyclic}}$$
Which proves claim 1.

2 points.

Claim 2: $XDBP$ is cyclic
Proof: Note that:
$$\underbrace{\angle CBF = \angle FBA}_{BF \text{ is the angle bisector of } \angle CBA} \overset{(AB) \parallel (CF)}{=} \angle BFC \overset{YD \equiv YF}{=} \angle YDF$$
hence, $(DY) \parallel (BC)$. Now, in one hand, we have:
$$\angle DPX + \angle FYP + \angle PFY = 0$$
$$\angle DPX + \angle ABC + \angle PFY = 0 \quad (DY) \parallel (BC)$$
So, we get:
$$\angle DPX = \angle CBA + \angle YFP$$

$$(1)$$

In other hand, we have:
$$\angle CBA = \angle DBX + \angle XBA + \angle CBE$$
$$= \angle DBX + \angle XCA + \angle CBF \quad (ABCX \text{ is cyclic})$$
$$= \angle DBX + \angle XFD + \angle DFC \quad (DCFX \text{ is cyclic})$$
$$= \angle DBX + \angle XFC$$
$$= \angle DBX + \angle PFY$$
So, we get:
$$\angle DBX = \angle CBA + \angle YFP$$

$$(2)$$

Combining the two previous results $((1)$ *and* $(2))$, we deduce that the quadrilateral $XDBP$ is cyclic.

2 points.

Notice that:
$$\angle DCZ = \angle DCF = \angle DXF = \angle DXP = \angle DBP = \angle DBZ$$
which implies $DCZB$ is cyclic, so $Z \in (BDC)$. Now, we have to show that $Z \in (PRY)$.

1 point.

Claim 3: $FRDZ$ is cyclic
Proof: We have:
$$\angle FRD = \angle PRD \overset{Claim\ 1}{=} \angle PXB \overset{Claim\ 2}{=} \angle PDB \overset{(DY) \parallel (BC)}{=} \angle CBD \overset{DCZB \text{ cyclic}}{=} \angle CZD = \angle FZD$$
which proves claim 3.

1 point.

<u>Claim 4:</u> $ZPRY$ cyclic
<u>Proof:</u> We have:
$$\angle PRZ = \angle PRD + \angle DRZ$$
$$= \angle PXB + \angle DFZ \quad (claim \; 1 \; and \; claim \; 3)$$
$$= \angle PDB + \angle DFY \quad (claim \; 2)$$
$$= \angle YDF + \angle DFY$$
$$= \angle FYD$$
$$= \angle PYD$$

which shows the desired claim.

2 points.

Before we finish, we show that $S \in (PRY)$. To do so, notice that:

$$\angle YSR = \angle ESR \stackrel{ETSR \; cyclic}{=} \angle ETR \stackrel{EDTR \; cyclic}{=} \angle EDR = \angle FDR \stackrel{Claim \; 3}{=} \angle FZR = \angle YZR \stackrel{Claim \; 4}{=} \angle YPR$$

Hence, we deduce that the points R, P, S and Y are concyclic and from what we did before, we find that the circumcircles of triangles $\triangle PRS$ and $\triangle BDC$ pass through the point Z.

1 point.

We finish the proof by taking the homothety centered at Z with ratio $\dfrac{ZC}{ZY}$, we get that C must be sent to Y and B must be sent to P (as $(BC) \parallel (PY)$), but Z belongs to (DBC) and (PRY), thus the circumcircles of triangles $\triangle DBC$ and $\triangle PRS$ are internally tangent which solves the problem.■

1 point.

Problem 4. Determine all sequences of positive real numbers $a_1, a_2, a_3, \ldots$, such that for each positive integer n the following equality holds:
$$a_n + \max(a_{n+1}, a_{n+2}) = \frac{1}{\min(a_n, a_{n+1})}.$$

(Ivan Novak)

Solution. The only constant sequence that is a solution is the sequence satisfying $a_n = \frac{1}{\sqrt{2}}$ for all n. Also, one can easily check that any sequence which contains two consecutive elements equal to $\frac{1}{\sqrt{2}}$ is constant. Now consider a nonconstant sequence $(a_n)_n$ satisfying the conditions.

Now suppose that $a_n \geqslant \frac{1}{\sqrt{2}}$ for some n. If $a_{n+1} > \frac{1}{\sqrt{2}}$, then the right hand side of the equality
$$a_n + \max(a_{n+1}, a_{n+2}) = \frac{1}{\min(a_n, a_{n+1})}$$

is at most $\sqrt{2}$, and the left hand side is greater than $\sqrt{2}$, which is a contradiction.

Thus, if $a_n \geqslant \frac{1}{\sqrt{2}}$, then $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$.

Now suppose that $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$. Then $a_n + a_{n+2} = \frac{1}{a_{n+1}}$. Furthermore, since $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$, one has $a_{n+3} \leqslant \frac{1}{\sqrt{2}}$ and from the recurrence for $n+1$ one obtains
$$a_{n+1} + a_{n+2} = \frac{1}{a_{n+1}},$$

so $a_n = a_{n+1} = \frac{1}{\sqrt{2}}$ and the sequence is constant, a contradiction.

We have thus proved that if $a_n \geqslant \frac{1}{\sqrt{2}}$, then $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$.

1 point.

Now suppose a_n, a_{n+1}, a_{n+2} are all $\leqslant \frac{1}{\sqrt{2}}$. Then the left hand side of the assertion is at most $\sqrt{2}$, and the right hand side is at least $\sqrt{2}$, with equality holding if $a_n = \frac{1}{\sqrt{2}} = a_{n+1}$, which would yield a constant sequence, a contradiction.

Thus, exactly every third element of the sequence is $\geqslant \frac{1}{\sqrt{2}}$, and the others are $\leqslant \frac{1}{\sqrt{2}}$.

Now if $a_n = \frac{1}{\sqrt{2}}$ and $a_{n+1} < \frac{1}{\sqrt{2}}$, then the right hand side is greater than $\sqrt{2}$, and the left hand side is at most $\sqrt{2}$, a contradiction. Thus, we may also assume that $\frac{1}{\sqrt{2}}$ does not occur in the sequence.

Now suppose $a_n > \frac{1}{\sqrt{2}}$. Then $a_{n+3} > \frac{1}{\sqrt{2}}$.

2 points.

If $a_{n+1} < a_{n+2}$, then from the assertion for $n+1$ one has
$$a_{n+1} + a_{n+3} = \frac{1}{a_{n+1}},$$

and from the assertion for $n+2$ one has
$$a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}.$$

From here, it follows that $a_{n+1} = a_{n+2}$, a contradiction.

If $a_{n+1} > a_{n+2}$, one has $a_{n+1} + a_{n+3} = \frac{1}{a_{n+2}}$ and $a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}$, so $a_{n+1} = a_{n+2}$, a contradiction.

Thus, $a_{n+1} = a_{n+2}$ whenever $a_n > \frac{1}{\sqrt{2}}$.

2 points.

Now we also have $a_n + a_{n+1} = \frac{1}{a_{n+1}}$ and $a_{n+2} + a_{n+3} = \frac{1}{a_{n+2}}$, so $a_n = a_{n+3}$ and the sequence is periodic with period 3, and of the form
$$\left(, \ldots, \frac{1}{x} - x, x, x, \ldots,\right)$$

for some $x < \frac{1}{\sqrt{2}}$, with possibly two initial members of the sequence being $< \frac{1}{\sqrt{2}}$ not being equal to $\frac{1}{x} - x$.

Now suppose that $a_n = \frac{1}{x} - x$, $a_{n+1} = x$ for some $x < \sqrt{2}$, and suppose that a_{n-1} exists.

Then $a_{n-1} + a_n = \frac{1}{a_{n-1}}$, so $a_{n-1} = x$. This means that even the first part of the sequence is periodic.

2 points.

In conclusion, the answer is all periodic sequences of length 3 of the form
$$\left(, \ldots, \frac{1}{x} - x, x, x, \ldots\right),$$

where $x < \frac{1}{\sqrt{2}}$, and the constant sequence $a_n = \frac{1}{\sqrt{2}}$.

1 point.

Now suppose $a_n \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+1} > \frac{1}{\sqrt{2}}$. Then $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$ and $a_n + a_{n+1} = \frac{1}{a_n}$

Now suppose that $a_n \leqslant \frac{1}{\sqrt{2}}$ and $a_{n+1} \leqslant \frac{1}{\sqrt{2}}$. Then the right hand side is at least $\sqrt{2}$, so $\max(a_{n+1}, a_{n+2})$ is at least $\frac{1}{\sqrt{2}}$, which means that $a_{n+2} \geqslant \frac{1}{\sqrt{2}}$, or $a_n = a_{n+1} = \frac{1}{\sqrt{2}}$ and $a_{n+2} \leqslant \frac{1}{\sqrt{2}}$.

2 points.