Problems and solutions of the 14th European Mathematical Cup

13th December 2025 - 21th December 2025

Junior Category

Problem 1. Let ω_1 and ω_2 be two circles intersecting at A and B. The common tangent, closer to A, of ω_1 and ω_2 touches ω_1 at P and ω_2 at Q. The tangent of ω_1 at A meets ω_2 at C, which is different from A, and the extension of AP meets QC at D. Let E be the centre of the circumcircle of triangle ABD. The lines AD and QE intersect at F. Prove that F lies on the circle with diameter PQ.

(Steve Vo Dinh)

Solution. It suffices to show that $\angle PFQ = 90^\circ$. Since $AE = DE$ (since E is the centre of the circumcircle of ABD), this is equivalent to QE being the perpendicular bisector of $\overline{AD}$, since we know that all points on the perpendicular bisector are equidistant from the endpoints of $\overline{AD}$. So, it is sufficient to prove $QA = QD$, and with this, we completely remove E from the diagram.

3 points.

For convenience, denote $\angle AQP = \alpha$, $\angle APQ = \beta$. Then:

$$\angle QAD = \alpha + \beta$$

from triangle APQ (since the exterior angle is equal to the sum of the two remaining interior angles).

1 point.

It remains to compute $\angle QDA = \angle ACD + \angle DAC$. We have

$$\angle ACD = \angle ACQ = \angle AQP = \alpha$$

from the tangent QP to ω_2 (by the Chord Tangent Lemma, also known as the Alternate Segment Theorem)

1 point.

On the other hand, $\angle DAC = 180^\circ - \angle PAC$ since P, A and D lie on the same line. However, since AC is tangent to ω_1, we can find $\angle CAB = \angle APB$ (again by the Chord Tangent Lemma).

Now, we can break the $180^\circ = \angle PAD = \angle PAB + \angle BAC + \angle CAD$ around point A, but we can also find the sum of internal angles in APB to be $\angle PAB + \angle BPA + \angle ABP = 180^\circ$, so by previous conclusion, we get that $\angle ABP = \angle CAD$.

3 points.

Lastly, $\angle ABP = \angle APQ = \beta$ since PQ is tangent to ω_1 (using the Chord Tangent Lemma one last time).

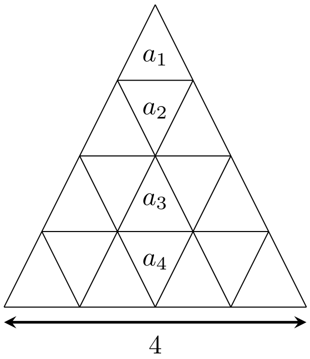
By looking at triangle CDA, we conclude that $\angle QDA = \angle ACD + \angle DAC$, so we have

$$\angle QDA = \angle ACD + \angle DAC = \alpha + \beta = \angle AQP + \angle APQ = \angle QAD$$

concluding $QA = QD$, as desired.

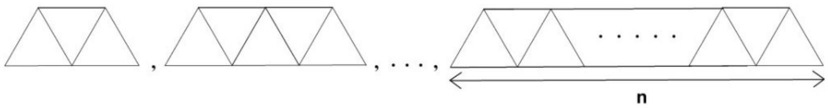
2 points.

Problem 2. Let n be a positive integer. Divide an equilateral triangle of side length n equilateral triangles of side length one. (example shown below for $n = 4$):



Label the small equilateral triangles through which one of the altitudes of the large equilateral triangle passes as $a_1, a_2, \ldots, a_n$ (see illustration above).

Let $f(i)$ denote the number of ways to tile the large equilateral triangle using exactly one of each



such that the triangle a_i is removed.

- If n is even, determine $f(2) + f(4) + \cdots + f(2k) + \cdots + f(n)$.
- Prove that $f(1) + f(2) + \cdots + f(n) \geqslant 2^{n-2}$, for all positive integers n.

(Karlo Jokoš)

First Solution. First Part

Suppose there is a tiling with triangle a_{2k} missing, we tile the triangle starting with the largest tile, we can put it at three places (left, right or down), and we are left with a triangle of side length $n-1$. We repeat that step repeatedly, choosing to put the tile left, right or down, until we are left with the triangle a_{2k}. But then we had to choose down $n-2k$ times, and left and right both the same number of times, so $\frac{n-1-(n-2k)}{2} = \frac{2k-1}{2} \notin \mathbb{N}$, which is impossible, hence $f(2k) = 0$ and the sum is 0.

Alternative proof of first part

Colour all the triangles pointing upward black and the others white. The full triangle has n more black triangles than white triangles. Each of the $n-1$ shapes we are filling the big triangle with has one more black than white triangles, so the last unfilled triangle must be black. This is not the case for even numbers, so $f(2k) = 0$ and the sum is 0.

Regardless of the way the student have proved the first part, they should be awarded **4 points.**

Second Part

Let S_n be the sum we are bounding, and assume that the inequality holds for $1, 2, \ldots, n$.

Let us prove the $n+1$ case, there are three places where we can put the largest tile, at the bottom of the big triangle, to the left or to the right.

In the first case there are S_n ways to tile the rest of the triangle, in the second case if we put the second largest tile on the right, we end up with a triangle of side length $n-1$ so there are at least S_{n-1} ways to tile the triangle in that case, and the same bound holds for the case where the largest tile is to the right. So we have $S_{n+1} \geq S_n + 2S_{n-1} \geq 2^{n-2} + 2 \cdot 2^{n-3} = 2^{n-1}$

So we have to check the base case for $n = 1, 2$ and we get

$$S_1 = 1 \geq \frac{1}{2}, \quad S_2 = 1 \geq 1$$

6 points.

Second Solution. Suppose there is a tiling with triangle a_k missing, we tile the triangle starting with the largest tile, we can put it at three places (left, right or down) and we are left with a triangle of side length $n-1$. We repeat that step repeatedly, choosing to put the tile left, right or down, until we are left with the triangle a_k. But then we had to choose down $n-k$ times, and left and right both the same number of times so $\frac{n-1-(n-k)}{2} = \frac{k-1}{2}$.
So we count the number of $n-1$-tuples with entries from $\{$down, left, right$\}$, such that there are $n-k$ downs, and $\frac{k-1}{2}$ lefts and rights. So $f(k) = \binom{n-1}{n-k}\binom{k-1}{\frac{k-1}{2}}$

6 points.

First Part

For even k we get that $f(k) = 0$ so if n is even the sum over the even numbers is 0.

1 point.

Second Part

The sum is equal to

$$\sum_{k \text{ odd}} \binom{n-1}{n-k}\binom{k-1}{\frac{k-1}{2}} \geq \sum_{k \text{ odd}} \binom{n-1}{n-k} = \frac{1}{2}\sum \binom{n-1}{n-k} = 2^{n-2}$$

3 points.

Problem 3. Determine the largest positive integer n for which there exist positive integers a and q such that

$$q^6 \le n \quad \text{and} \quad \left|\sqrt{2} - \frac{a}{q}\right| \le \frac{1}{\sqrt{n}}.$$

(Miroslav Marinov)

First Solution. Assume $q \ge 2$. Then

$$\left|\sqrt{2} - \frac{a}{q}\right| \le \frac{1}{\sqrt{n}} \le \frac{1}{q^3} \le \frac{1}{8}.$$

In particular,

$$\frac{a}{q} \le \sqrt{2} + \frac{1}{8},$$

so

$$\sqrt{2} + \frac{a}{q} \le 2\sqrt{2} + \frac{1}{8} < 3.$$

On the other hand, using the fact that $|2q^2 - a^2| \ge 1$ (since $\sqrt{2}$ is irrational and hence $2q^2 - a^2 \ne 0$), we obtain

$$\frac{1}{q^2} \le \frac{|2q^2 - a^2|}{q^2} = \left|\sqrt{2} - \frac{a}{q}\right|\left|\sqrt{2} + \frac{a}{q}\right| < \frac{3}{q^3}.$$

Therefore $q < 3$. Since q is a positive integer, it remains to consider only $q = 1$ and $q = 2$.

Case 1: $q = 1$**.** We would need

$$\left|\sqrt{2} - a\right| \le \frac{1}{8}.$$

However, since $1.4 < \sqrt{2} < 1.5$, the distance from $\sqrt{2}$ to the nearest integer is greater than 0.4, which is larger than 1/8. (Alternatively, we can proceed directly to $q = 2$ by arguing that if we find a possible n with $q = 2$, then the case $q = 1$ is automatically covered by the case $q = 2$, since any integer can be written as a fraction with denominator 2.)

Case 2: $q = 2$**.** For $n \ge 2^6$ we must have

$$\left|\sqrt{2} - \frac{a}{2}\right| \le \frac{1}{8}.$$

If $a \ne 3$, then

$$\left|\sqrt{2} - \frac{a}{2}\right| \ge \left|\sqrt{2} - 1\right| > \frac{1}{8},$$

so the only possible value is $a = 3$.

7 points.

It remains to compute the largest n such that

$$\left|\sqrt{2} - \frac{3}{2}\right| \le \frac{1}{\sqrt{n}}.$$

Equivalently,

$$n \le \frac{1}{\left(\frac{3}{2} - \sqrt{2}\right)^2} = \left(\frac{3}{2} + \sqrt{2}\right)^2 = 68 + 48\sqrt{2}.$$

Since $68 + 48\sqrt{2} < 136$ and $68 + 48\sqrt{2} > 135$, the largest possible integer value is

$$\boxed{135}.$$

3 points.

Second Solution. (Alternative proof of $q \le 2$)
We will find the largest integer n when $q \leqslant 2$ in the same order as in the first solution.

3 points.

We have

$$\begin{aligned} &\left|\sqrt{2} - \frac{a}{q}\right| \le \frac{1}{n} \le \frac{1}{q^3} \\ \implies &|\sqrt{2}q^3 - aq^2| \le 1 \\ \implies &\sqrt{2}q - \frac{1}{q^2} \le a \le \sqrt{2}q + \frac{1}{q^2} \end{aligned}$$

Squaring gives

$$2q^2 - \frac{2\sqrt{2}}{q} + \frac{1}{q^4} \leqslant a^2 \leqslant 2q^2 + \frac{2\sqrt{2}}{q} + \frac{1}{q^4}$$

Now assume for contradiction that $q \geqslant 3$ and notice that we have $\frac{2\sqrt{2}}{q} - \frac{1}{q^4} < \frac{2\sqrt{2}}{q} + \frac{1}{q^4} \le \frac{2\sqrt{2}}{3} + \frac{1}{81} < \frac{29}{30} + \frac{1}{30} = 1$. Hence $2q^2 - 1 < a^2 < 2q^2 + 1$. Since a and q are integers, we must have $a^2 = 2q^2$, but this is impossible, as $\sqrt{2}$ is irrational.

7 points.

Problem 4. Find all positive integers n with the following property:
For every positive integer d which divides n, there exists a positive integer k which divides n such that

$$d + n \mid dn + k.$$

(Ivan Novak)

Solution. Note that $n = 1$ is obviously a solution. Now consider $n > 1$.
Taking $d = n$ yields $2n \mid n^2 + k$, which gives us $k = n$ and $2n \mid n^2 + n$, which implies that n must be odd.
We'll now prove that n must be a prime power.
Note that $d \mid d + n$ for any divisor d of n, so if $d + n \mid dn + k$, it follows that $d \mid k$.
We can rewrite the condition as

$$1 + \frac{n}{d} \mid n + \frac{k}{d}.$$

Now note that $n \equiv -d \pmod{1 + \frac{n}{d}}$, so the condition is equivalent to

$$1 + \frac{n}{d} \mid \frac{k}{d} - d.$$

If $\frac{n}{d} \geqslant d$, then $|\frac{k}{d} - d| \leqslant \frac{n}{d} < 1 + \frac{n}{d}$, so we must have $\frac{k}{d} = d$, or $k = d^2$.
This means that for any divisor $d \leqslant \sqrt{n}$ of n, d^2 also divides n.

2 points.

Now suppose that n is not a power of a prime. Let $n = ab$ with $a > b$ coprime and greater than 1. Then b^2 divides $n = ab$, but then b divides a, a contradiction. Thus, n is a power of a prime.

1 point.

Let $n = p^m$ for some positive integer m.
Let $d = p^j$ with $m \geqslant j > \frac{m}{2}$. Then there exists p^ℓ with $j \leqslant \ell \leqslant m$ such that

$$p^j + p^m \mid p^{m+j} + p^\ell,$$

which is equivalent to

$$p^{m-j} + 1 \mid p^{m-\ell+j} + 1.$$

Recall the well known fact that if $a \geqslant 2$, x and y are positive integers such that $a^x + 1$ divides $a^y + 1$, then $\frac{y}{x}$ is an odd integer. Hence $m - \ell + j$ must be an odd number times $m - j$, i.e. there exists a nonnegative integer r with

$$m - \ell + j = (2r + 1)(m - j),$$

or equivalently

$$2r(m - j) = 2j - \ell \leqslant j.$$

Now take $j = \lfloor m/2 \rfloor + 1$. We get $2r(\lceil m/2 \rceil - 1) = 2\lfloor m/2 \rfloor + 2 - \ell$. For $r = 0$ this doesn't work as then we'd have $\ell > m$. For $r > 0$ the left hand side is at least $m - 2$, and the right hand side is at most $m/2 + 1$, so we have $m - 2 \leqslant m/2 + 1$, or $m \leqslant 6$.

4 points.

For $m = 5$, we must have

$$2r(5 - j) = 2j - \ell.$$

Take $j = 3$. We get $4r = 6 - \ell \leqslant 3$, impossible. Thus, $m = 5$ is also not a solution.

1 point.

For $m \in \{1, 2, 3, 4, 6\}$, we can easily check that for any $j > m/2$ we can find r and ℓ satisfying the condition

$$2r(m - j) = 2j - \ell.$$

Thus, the set of solutions is

$$\{p^m \mid p \text{ an odd prime }, m \in \{0, 1, 2, 3, 4, 6\}\}.$$

2 points.