13^{TH} European Mathematical Cup

14^{th} December 2024 - 22^{nd} December 2024

Senior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problem 1. We call a pair of distinct numbers (a, b) a *binary pair* if $ab + 1$ is a power of two. Given a set S of n positive integers, what is the maximum possible number of binary pairs in S?

(Oleksii Masalitin)

Problem 2. Let n be a positive integer. The numbers $1, 2, \ldots, 2n + 1$ are arranged in a circle in that order, and some of them are *marked*.

We define, for each k such that $1 \leqslant k \leqslant 2n + 1$, the interval I_k to be the closed circular interval starting at k and ending at $k + n$ (taking remainders modulo $2n + 1$ if $k + n > 2n + 1$). We call an interval I_k *magical* if it contains strictly more than half of all the marked elements.

Prove that the following two statements are equivalent:

1. At least $n + 1$ of the intervals $I_1, I_2, \ldots, I_{2n+1}$ are magical.
2. The number of marked numbers is odd.

(Andrei Constantinescu)

Problem 3. Let ω be a semicircle with diameter $\overline{AB}$ and let M be the midpoint of $\overline{AB}$. Let X, Y be points in the same half-plane as ω with respect to the line AB such that $AMXY$ is a parallelogram. Let I be the incenter of the triangle MXY. Lines MX, MY intersect ω in points C, D respectively. Let T be the intersection of AC and BD. The line MT intersects XY in E. If P is the intersection of EI and AB, and Q is the projection of E onto the line AB, show that M is the midpoint of $\overline{PQ}$.

(Michal Pecho)

Problem 4. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + yf(x)) = xf(1 + y)$$

for all $x, y \in \mathbb{R}^+$.

Remark. We denote by $\mathbb{R}^+$ the set of all positive real numbers. *(Ioannis Galamatis)*

Time: 240 minutes. *Each problem is worth 10 points.*

*The use of calculators or any other instruments except rulers and compasses is not permitted.
Cheating in any form is strictly prohibited. Sharing or posting the problems or solutions before the end of the competition (22nd of December) to anyone who might participate is strictly prohibited.*