Problems and solutions of the 13th European Mathematical Cup

14th December 2024 - 22nd December 2024

Senior Category

Problem 1. We call a pair of distinct numbers (a, b) a *binary pair* if $ab + 1$ is a power of two. Given a set S of n positive integers, what is the maximum possible number of binary pairs in S?

(Oleksii Masalitin)

First Solution. The answer is $n - 1$, achieved by choosing $2^k - 1$ for $1 \leq k \leq n$. One can then easily see that $1, 2^k - 1$ makes a binary pair for $n \geq k > 1$. For the bound, make a graph G with vertices $a_1, \dots a_n$, and connect (a, b) if a, b makes a binary pair.

2 points.

The key is the following:

Claim 1. *G does not contain a cycle.*

Proof. Assume otherwise and suppose:

$$\begin{aligned} x_1x_2 + 1 &= 2^{a_1} \\ x_2x_3 + 1 &= 2^{a_2} \\ &\dots \\ x_nx_1 + 1 &= 2^{a_n} \end{aligned}$$

for positive integers $a_1, \dots a_n$ and $n \geqslant 3$. WLOG let a_1 be the greatest (not necessarily unique). Notice that all x_i are odd, this is just parity. Focus on the following three equations:

$$x_nx_1 + 1 = 2^{a_n} \tag{1}$$
$$x_1x_2 + 1 = 2^{a_1} \tag{2}$$
$$x_2x_3 + 1 = 2^{a_2} \tag{3}$$

Subtracting (1) from (2) and (3) from (2) results in:

$$x_1(x_2 - x_n) = 2^{a_n}(2^{a_1 - a_n} - 1) \implies x_1 \mid 2^{a_1 - a_n} - 1 \text{ as } x_1 \text{ is odd}$$
$$x_2(x_1 - x_3) = 2^{a_2}(2^{a_1 - a_2} - 1) \implies x_2 \mid 2^{a_1 - a_2} - 1 \text{ as } x_2 \text{ is odd}$$

3 points.

The bound $x_1x_2 + 1 < (x_nx_1 + 1)(x_2x_3 + 1)$ can be verified by simple expansion. In turn, this gives $a_2 + a_n > a_1$

3 points.

Finally:

$$2^{a_1} - 1 = x_1x_2 \mid (2^{a_1 - a_2} - 1)(2^{a_1 - a_n} - 1) < 2^{a_1} - 2^{a_1 - a_2} - 2^{a_1 - a_n} + 1 \leqslant 2^{a_1} - 1$$

As $(2^{a_1 - a_2} - 1)(2^{a_1 - a_n} - 1)$ is nonnegative, the divisibility can hold only if it is equal to 0. This would give $a_1 = a_n$ or $a_1 = a_2$, contradicting $x_n \neq x_2$ and $x_1 \neq x_3$ respectively.

□

As a graph with n vertices not containing cycles can have at most $n - 1$ edges, the proof of the bound is finished.

2 points.

Second Solution. Here we present an approach using Zsigmondy's theorem. Assume that $T = \{b_1, \dots b_k\}$ is the set of all indices with the maximal value of a_i, call it M. Observe that a_j-s in T must not be consecutive, otherwise they would violate the distinct x_i condition. Now we can multiply the equations to get the following:

$$\prod_{i \in T}(2^M - 1) \mid \prod_{i \notin T}(2^{a_i} - 1)$$

2 points.

We get a contradiction with Zsigmondy's theorem as $2^M - 1$ has a primitive prime divisor not dividing $2^n - 1$ for $n < M$, except for the case $M = 6$

4 points.

. Assume $x_i x_{i+1} = 63$ for some i.

- if $x_i = 63$, then $x_{i+2} = 63$ as 2^6 is the greatest possible value, contradicting distinctness.
- if $x_i = 21$, one similarly gets $x_{i+2} = 21$, again a contradiction.
- the rest of the cases are similar and bruteforce

2 points.

Third Solution. Again, the construction and the comment is the same as in the first solution. For the bound, we claim the following:
If $(a, b), (a, c)$ are two binary pairs with $a > b, c$, then $b = c$.

1 point.

Proof. WLOG assume $b \geq c$. As $ac + 1$ and $ab + 1$ are powers of 2, we have:

$$ac + 1 \mid ab + 1$$
$$ac + 1 \mid a(b - c)$$
$$ac + 1 \mid b - c$$

4 points.

But, if $b > c$, then

$$0 < b - c < b < a < ac + 1$$

2 points.

, contradiction. □

Now, all elements of S except the smallest can be the larger element in at most one binary pair, giving the bound $n - 1$

1 point.

Comment: One can show no cycles of odd length with little effort: All powers of 2 involved are at least 4. Now subtract one and multiply all equations together, and finish by mod 4.

Problem 2. Let n be a positive integer. The numbers $1, 2, \ldots, 2n+1$ are arranged in a circle in that order, and some of them are *marked*.

We define, for each k such that $1 \leqslant k \leqslant 2n+1$, the interval I_k to be the closed circular interval starting at k and ending at $k+n$ (taking remainders modulo $2n+1$ if $k+n > 2n+1$). We call an interval I_k *magical* if it contains strictly more than half of all the marked elements.

Prove that the following two statements are equivalent:

1. At least $n+1$ of the intervals $I_1, I_2, \ldots, I_{2n+1}$ are magical.
2. The number of marked numbers is odd.

(Andrei Constantinescu)

First Solution. Let S be the set containing all the marked numbers and $S_i = S \cap I_i$. Note that $S_i \cap S_{i+n} = \emptyset$ or $\{i+n\}$. So for each i we have that

$$|S_{i+n}| = \begin{cases} |S| - |S_i| + 1, & \text{if } i+n \in S \\ |S| - |S_i|, & \text{otherwise.} \end{cases}$$

2 points.

Suppose $|S|$ is odd. For each interval, I_i, that isn't magical we have that $|S_i| < \frac{|S|}{2}$ (since the equality can't hold) hence $|S_{i+n}| \geq |S| - |S_i| > |S| - \frac{|S|}{2} = \frac{|S|}{2}$ so I_{i+n} is magical. So for each non magical interval we can find a unique magical one, therefor we must have at least $n+1$ magical intervals.

4 points.

Now suppose $|S|$ is even. For each magical interval I_i, we have that $|S_i| > \frac{|S|}{2}$ hence $|S_{i+n}| \leq |S| - |S_i| + 1 < |S| - \frac{|S|}{2} + 1 = \frac{|S|}{2} + 1 \implies |S_{i+n}| \leq \frac{|S|}{2}$ so I_{i+n} is not magical. Since for each magical interval we can find a unique non magical one, there must be at least $n+1$ non magical intervals, so less that $n+1$ magical ones.

4 points.

Second Solution. Represent the remainders modulo $2k+1$ in a circle in ascending order. For the rest of the solution, *t-good* means the interval $[t, t+k]$ is magical, and the set being very good means it satisfies property (1). Label the vertices of the graph as $(0, 1, 2, \ldots, 2k)$. If S is t-good, draw an edge between t and $t+k$ (taken modulo $2k+1$). Now we prove both directions:

- Let $|S| = 2l$ be **very good**. The critical observation is the following: there is a node with degree 2. Since S is **very good**, there are at least $k+1$ edges. Thus, the total sum of degrees is at least $2k+2$.

  1 point.

  By the pigeonhole principle, there is a node with degree 2. Let a be the value of this node.

  1 point.

  By definition, the intervals $[a, a+k]$ and $[a-k, a]$ each contain more than half of the elements of S, i.e., at least $l+1$ elements each. These two intervals share exactly one element.

  2 points.

  Thus, the total number of distinct elements of S in these intervals is at least:

  $$2l+2 \quad (\text{if } a \notin S), \quad \text{or} \quad 2l+1 \quad (\text{if } a \in S).$$

  This contradicts $|S| = 2l$. Hence, if S is **very good**, S must have an odd number of elements.

  1 point.

- Now let $|S| = 2l-1$. A similar lemma applies: every node has degree at least 1. Fix a value x. The intervals $[x-k, x]$ and $[x, x+k]$ cover the entire residue class modulo $2k+1$ and share exactly one element. Now we split into two cases:
  - If $x \in S$, the remaining elements of S are in two disjoint intervals $[x-k, x-1]$ and $[x+1, x+k]$. By the pigeonhole principle, one of these intervals contains at least $l-1$ elements. Adding x creates a good interval with one of its endpoints at x, so x has degree at least 1.
  - If $x \notin S$, the elements of S are in two disjoint intervals $[x-k, x-1]$ and $[x+1, x+k]$. By the pigeonhole principle, one of these intervals contains at least l elements, so the conclusion is the same as above.

  In either case, x has degree at least 1.

  3 points.

  Similarly, every node has degree at least 1, so the total sum of degrees is at least $2k+1$. By the handshake lemma, the total sum of degrees is at least $2k+2$. This gives at least $k+1$ edges. Thus, S is indeed x **very good**.

  2 points.

Problem 3. Let ω be a semicircle with diameter $\overline{AB}$ and let M be the midpoint of $\overline{AB}$. Let X, Y be the points in the same half-plane as ω with respect to the line AB such that $AMXY$ is a parallelogram. Let I be the incenter of the triangle MXY. Lines MX, MY intersect ω in points C, D respectively. Let T be the intersection of AC and BD. The line MT intersects XY in E. If P is the intersection of EI and AB, and Q is the projection of E onto the line AB, show that M is the midpoint of $\overline{PQ}$.

(Michal Pecho)

Solution. We will work in the context of a triangle MXY. The solution is in two parts:

Claim 2. *E is the M-excircle touch point with XY*

We present two proofs:

Proof. We first claim that TCD is tangent to both MX and MY. Observe:

$$\angle MCT = \angle MCA = \angle CAM = \angle CAB = \angle CDB = \angle CDT$$

where the angles are directed. This shows that MX is tangent to DCT. Analogous proof gives MY tangent to CDT.

2 points.

Now we claim that the tangent to DCT at T is parallel to XY. The cleanest way is through negative inversion in T fixing the circle with diameter AB. This sends (CDT) into AB and fixes the tangent. The tangency is preserved, so the two lines are parallel, as desired.

1 point.

If the mentioned tangent meets MX and MY at R, S respectively, we have shown that CDT is M-excircle in MRS. Consider the homothety centred at M that maps RS to XY. It also maps T into E, but it also sends the excircle of MRS into the excircle of MXY, hence sending the M-touchpoint in MRS (i.e. T) into E, which is what we wanted to show.

3 points.

□

Proof. Let U, V be points of XY such that U, X, Y, V lie on XY in that order, and $UX = XM, VY = YM$. Also let R, S be the intersections of XY with AC, BD respectively (note that these do not correspond to R, S in the previous proof). Easy angle chase gives:

$$\begin{aligned}\angle YRT = \angle XRC &= \angle MXY - \angle MCA \\ &= \angle BMC - \angle MCA \\ &= \frac{1}{2}\angle YXM \\ &= \angle XUM\end{aligned}$$

, so $MU \parallel TR$. Similarly $MV \parallel TS$. Now the triangles TRS and MUV are homothetic

4 points.

, with E being the homothety center. The idea is that we can now express all the relevant lengths in terms of MXY. Let $a = XY, b = YM, c = MX$. Compute:

$$\begin{aligned}XR = XC &= a - c \\ YS = YD &= a - b \\ RS = XY - XR - SY &= b + c - a\end{aligned}$$

From Thales,

$$\frac{ER}{ES} = \frac{RU}{SV} = \frac{a}{a} = 1$$

and finally $XE = XR + \frac{1}{2}RS = a - c + \frac{b+c-a}{2} = \frac{a+b-c}{2}$, which is precisely the distance from X to M-extouchpoint.

2 points.

□

Having established that, let Z be the midpoint of M-altitude in MXY and N be its foot. It is well-known that Z, I, E are collinear, one can get that for example with homothety mapping the incircle to M-excircle

2 points.

. We are now done, check that $PM = NE$ from congruent MPZ and ZNE and $MQ = NE$ from the rectangle.

2 points.

Problem 4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + yf(x)) = xf(1 + y)$$

for all $x, y \in \mathbb{R}^+$.

Remark. We denote by $\mathbb{R}^+$ the set of all positive real numbers. *(Ioannis Galamatis)*

First Solution. The function $f(x) = x$ for all $x \in \mathbb{R}^+$ ssatisfies the condition, and we will show it is the only such function.

Firstly, note that if $f(x) \neq 1$ and we have that either $x > 1, f(x) < 1$ or $x < 1, f(x) > 1$, plugging in x and

$$y = \frac{1 - x}{f(x) - 1} > 0$$

gives that $x = 1$, a contradiction. Therefore, for all $x < 1$ we have $f(x) \leqslant 1$ and for all $x > 1$ we have $f(x) \geqslant 1$.

1 point.

Now, assume that $f(t) < t$ for some $t \in \mathbb{R}^+$, and take

$$x = f(t), y = \frac{t - f(t)}{f(f(t))}$$

in the starting equation. If we denote $s = \frac{t - f(t)}{f(f(t))}$ this gives

$$f(t) = f(t)f(1 + s) \implies f(1 + s) = 1$$

as the left-hand side and the $f(t)$ term on the right cancel out. If we now plug in

$$x = 1 + \frac{s}{2}, y = \frac{s}{2f\left(1 + \frac{s}{2}\right)}$$

we obtain that

$$1 = f(1 + s) = \left(1 + \frac{s}{2}\right) \cdot f(1 + y) \geqslant 1 + \frac{s}{2}$$

which is a contradiction.

Therefore, $f(x) \geqslant x$ for all $x \in \mathbb{R}^+$.

2 points.

If we apply this inequality to the left-hand side, we obtain that

$$xf(1 + y) \geqslant x + yf(x) \iff \frac{f(x)}{x} \leqslant \frac{f(y + 1) - 1}{y}$$

for all $x, y \in \mathbb{R}^+$. Plugging in $y = 1$, we obtain that $f(x)/x$ is bounded by $f(2) - 1$ for all $x \in \mathbb{R}^+$, and we already know it's always at least 1. Define

$$C = \limsup_{x \to \infty} \frac{f(x)}{x}.$$

We easily obtain that $\frac{f(y')-1}{y'-1} \geqslant C$ for all $y' = y + 1 \in \langle 1, \infty \rangle$, which rearranges to $f(y') \geqslant Cy' + 1 - C$ for all $y' > 1$. Additionally, by definition of C, for all $\varepsilon > 0$ there exists some $T_\varepsilon > 0$ such that $x > T_\varepsilon \implies f(x) \leqslant (C + \varepsilon)x$. Now, take some $x, y > \max\{1, T_\varepsilon\}$ and notice that $x + yf(x) > 1$ and $y + 1 > T_\varepsilon$ implies that we have

$$Cx + C^2xy + (1 - C)(y + 1) \leqslant f(x + yf(x)) = xf(y + 1) \leqslant x(C + \varepsilon)(y + 1) = Cxy + Cx + \varepsilon xy + \varepsilon x$$

which rearranges to

$$(C^2 - C - \varepsilon)y \leqslant \frac{(C - 1)(y + 1)}{x} + \varepsilon$$

for all x, y large enough. By fixing y and taking $x \to \infty$ we see that $C^2 - C \leqslant \varepsilon + \frac{\varepsilon}{y} < 2\varepsilon$ and as ε was arbitrary, we have that $C^2 - C = 0$ so $C = 1$.

5 points.

To finish, note that there now exists, for every $\varepsilon > 0$, a T_ε such that $y + 1 > T_\varepsilon$ implies $f(y + 1) \leqslant (1 + \varepsilon)(y + 1)$ and inserting this y into the inequality we earlier obtained gives that

$$\frac{f(x)}{x} \leqslant \frac{y + \varepsilon y + \varepsilon}{y} = (1 + \varepsilon) + \frac{\varepsilon}{y} < 1 + 2\varepsilon$$

for every $x \in \mathbb{R}^+$ and as ε is arbitrary, we obtain $f(x) \leqslant x$ for all $x \in \mathbb{R}^+$ and we are done.

2 points.

Second Solution. We shall firstly prove the following lemma about the behavior of f.

Lema 1. *The function f is non-decreasing.*

Proof. Assume on the contrary, there are $a > b$ such that $f(a) < f(b)$. Then, $c = \frac{a-b}{f(b)-f(a)}$ is positive. Now, plugging $(x, y) = (a, c)$ yields

$$f(\frac{af(b) - bf(a)}{f(b) - f(a)}) = f(a + \frac{a - b}{f(b) - f(a)}f(a)) = af(1 + c)$$

Plugging $(x, y) = (b, c)$ yields;

$$f(\frac{af(b) - bf(a)}{f(b) - f(a)}) = f(b + \frac{a - b}{f(b) - f(a)}f(b)) = bf(1 + c)$$

Yielding, $a = b$, a contradiction. This completes our proof.

3 points.

□

Now, it is easy to find that f is *surjective*, indeed, $f(x/f(2) + f(x/f(2)) = x$.

1 point.

Thus, f would be *continuous* [1].

2 points.

Hence, $f(x) = \lim_{y\to 0^+} f(x + yf(x)) = x \lim_{y\to 0^+} f(1 + y) = xf(1)$.

3 points.

That is, $f(x) = Cx$, for some $C > 0$. Hence, $C(x + C^2xy) = Cx(1 + y)$, yielding $C = 1$. It is easy to verify that $f(x) = x$ indeed satisfies the statement of the problem.

1 point.

Third Solution. For $x > 1$ we have $f(x) \geq 1$ otherwise plugging in $y = \frac{1-x}{f(x)-1}$ gives us a contradiction.
Applying this to the *RHS* of the original equation we get $f(x + yf(x)) \geq x$.
Putting $x = s - \epsilon, y = \frac{\epsilon}{f(s-\epsilon)}, \epsilon \to 0$ we get $f(s) \geq s$ for all $s \in \mathbb{R}^+$.

3 points.

Applying this to the *LHS* of the original equation, yielding;

$$f(y + 1) \geq 1 + y\frac{f(x)}{x} \qquad \forall x, y \in \mathbb{R}^+$$

1 point.

If there exist $c > 1$ such that $\frac{f(c)}{c} > 1$. Put $K = \frac{f(c)}{c}$.

Claim 3. $\quad f(y + 1) \geq yK^n \qquad \forall n \in \mathbb{N}, \forall y \in \mathbb{R}^+$.

Proof. We shall prove it through induction. The base is clear. Putting $(x, y) = (c, y)$ into original equation and assuming $f(y + 1) \geq yK^n$ for all y and fixed n:

$$cf(y + 1) = f(c + yf(c)) \geq (c + yf(c) - 1)K^n \geq yf(c)K^n$$

gives us $f(y + 1) \geq yK^n \implies f(y + 1) \geq yK^{n+1}$.

4 points.

□

Back to our problem, fixing y and using the claim letting $n \to \infty$ we get that such K can't exist, that is $f(x) \leq x$ for $x > 1$. Since we already proved $f(x) \geq x$ we have $f(x) = x$ for $x > 1$, but returning to $f(y+1) = y+1 \geq 1+y\frac{f(x)}{x}$ gives us $f(x) = x$ for all x. It is easy to check that $f(x) = x$ indeed satisfies the original equation.

2 points.

---

[1] For sake of completeness, in the following we shall provide the *outline* of the proof of this claim: Since f is non-deceasing, for an arbitrary positive a, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$ exist and $\lim_{x\to a^-} f(x) \leq \lim_{x\to a^+} f(x)$. Now, we prove these two limits are equal. Assume for contradiction, $b = \lim_{x\to a^-} f(x) < \lim_{x\to a^+} f(x) = c$. Thus, for all $x < a$ we have $f(x) \leq b$, for all $x > a$, we have $f(x) \geq c$. Hence the image of function is a subset of $(0,\ b) \cup (c,\ +\infty) \cup \{f(a)\}$. This, can not be the whole $\mathbb{R}^+$. The derived contradiction, completes our proof.