Problems and solutions of the 13th European Mathematical Cup

14th December 2024 - 22nd December 2024

Junior Category

Problem 1. Wiske wrote a 2024-digit positive integer on the blackboard. In each round of the game she erases the last digit of the integer, let this digit be d, and writes down the sum of the remaining number and $2d$ in place of the old number. She repeats the same steps with the newly obtained number. After a certain number of rounds, Wiske found that the new number obtained was the same as the number in the last round and she stopped the game. What is the smallest possible 2024-digit integer that Wiske started with in this game?

(Kai Chen)

Solution. Firstly, we claim that the last number Wiske obtained is 19.

In the last round, let the last digit of the number be d where $0 \leqslant d \leqslant 9$, and the remaining digits form an integer x. The number at the beginning of this round is then $10x + d$, and the new number obtained in this round is $x + 2d$. Since the two numbers are the same, $10x + d = x + 2d$, i.e., $9x = d$.
Because $0 \leqslant d \leqslant 9$ and we cannot have $x = d = 0$ because all newly written numbers are positive, the only solution is $x = 1$ and $d = 9$. The last number is then 19.

2 points.

Secondly, we claim that the numbers in all previous rounds are divisible by 19. From $2(10x + d) = 19x + (x + 2d)$, it follows that $2(10x + d) \equiv x + 2d \ (mod\ 19)$. Since the last number is 19, it can be concluded by reverse induction that the numbers in all rounds of the game are divisible by 19.

3 points.

From the same induction we get that the number Wiske started with being divisible by 19 is a sufficient condition as well.

1 point.

Finally, our goal is to find the smallest 2024-digit number which is divisible by 19 because the sequence of the numbers in all rounds is strictly descending:

$$10x + d > x + 2d, \text{ if } x > 1.$$

Per the Fermat's Little Theorem, we get $10^{18} \equiv 1 \ (mod\ 19)$. We have,

$10^{2023} \equiv 10^7 \ (mod\ 19)$, because $2023 = 112 \times 18 + 7$.
$10^7 = 100^3 \times 10 \equiv 5^3 \times 10 = 1250 \equiv 15 \ (mod\ 19)$.
$10^{2023} + 4 \equiv 0 \ (mod\ 19)$.

3 points.

Thus, $10^{2023} + 4$ is the smallest 2024-digit number which is divisible by 19. It is the smallest possible 2024-digit integer that Wiske started with in the game.

1 point.

Problem 2. Let X be the largest possible value of the expression

$$\min\{bc, 2-a^2\} + \min\{ac, 2-b^2\} + \min\{ab, 2-c^2\},$$

where a, b and c are positive real numbers. Similarly, let Y be the smallest possible value of the expression

$$\max\{a^2, 2-bc\} + \max\{b^2, 2-ac\} + \max\{c^2, 2-ab\},$$

where a, b and c are positive real numbers. Prove that $X = Y$. *(Ognjen Tešić)*

First Solution. Observe that $\min\{bc, 2-a^2\} = 2 + \min\{bc-2, -a^2\} = 2 - \max\{-(bc-2), a^2\} = 2 - \max\{2-bc, a^2\}$, so $X = Y$ is equivalent to

$$\begin{aligned}
&\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = \max\left\{\sum_{cyc}\min\{bc, 2-a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = \max\left\{\sum_{cyc} 2-\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 6 + \max\left\{-\sum_{cyc}\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 6 - \min\left\{\sum_{cyc}\max\{2-bc, a^2\}\right\}\\
\iff &\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} = 3
\end{aligned}$$

3 points.

For $a=b=c=1$ we get that $\min\left\{\sum_{cyc}\max\{a^2, 2-bc\}\right\} \le 3$, so we need to show that

$$\sum_{cyc}\max\{a^2, 2-bc\} \ge 3$$

1 point.

but since $\max\{x,y\} \ge \frac{x+y}{2}$ we have that

2 points.

$$\sum_{cyc}\max\{a^2, 2-bc\} \ge \sum_{cyc}\frac{a^2+2-bc}{2} = 3 + \frac{1}{2}\left(\sum_{cyc}a^2 - \sum_{cyc}bc\right) = 3 + \frac{1}{2}\left(\sum_{cyc}\frac{b^2+c^2}{2} - \sum_{cyc}bc\right) \ge 3 + \frac{1}{2}\left(\sum_{cyc}bc - \sum_{cyc}bc\right) = 3$$

4 points.

Second Solution. An alternative proof of the last inequality. Suppose, for the sake of contradiction, that

$$\sum_{cyc}\max\{a^2, 2-bc\} < 3$$

but since $\min\{a^2, 2-bc\} \le \max\{a^2, 2-bc\}$ we have that

$$\sum_{cyc}\min\{a^2, 2-bc\} \le \sum_{cyc}\max\{a^2, 2-bc\} < 3$$

by adding the two inequalities we get

$$\sum_{cyc}\max\{a^2, 2-bc\} + \max\{a^2, 2-bc\} < 6$$

1 point.

now we use the fact that $\min\{x,y\} + \max\{x,y\} = x+y$

1 point.

to get that

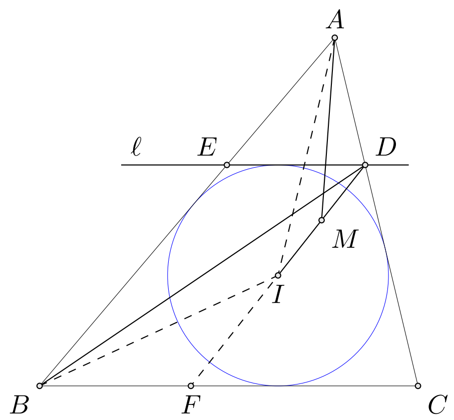
$$\sum_{cyc} a^2 + 2 - bc < 6,$$

contradiction by AG inequality as in the First Solution.

4 points.

Problem 3. Let ABC be a triangle with incenter I and incircle ω. Let ℓ be the tangent to ω parallel to BC and distinct from BC. Let D be the intersection of ℓ and AC, and let M be the midpoint of $\overline{ID}$. Prove that $\angle AMD = \angle DBC$.

(Weihua Wang)



First Solution. Let E be the intersection point of line ℓ with AB, and let F be the intersection point of DI with BC. Since $\ell \parallel BC$ and ℓ is tangent to circle ω, we have

$$\angle CDI = \angle EDI = \angle CFI = 90^\circ - \frac{C}{2},$$

$$\angle ADI = \angle BFI = 90^\circ + \frac{C}{2} = \angle AIB.$$

2 points.

Noting that $\angle DAI = \angle BAI$ and $\angle ABI = \angle FBI$, we obtain $\triangle ADI \sim \triangle AIB \sim \triangle IFB$. Therefore,

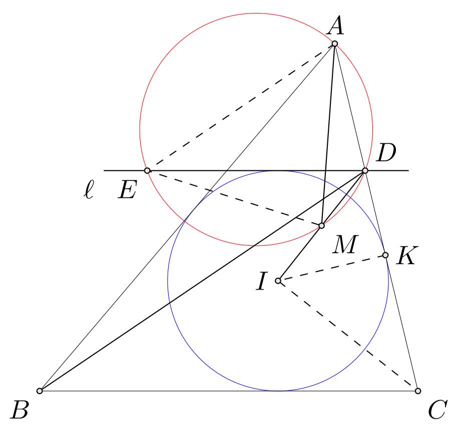
2 points.

$$\frac{AD}{DI} = \frac{IF}{FB} \Rightarrow \frac{AD}{DM} = \frac{2AD}{DI} = \frac{2IF}{BF} = \frac{DF}{BF}.$$

4 points.

Thus, $\triangle AMD \sim \triangle DBF$, so $\angle AMD = \angle DBC$.

2 points.



Second Solution. Let $BC = a$, $CA = b$, $AB = c$, and let the inradius of $\triangle ABC$ be r. Suppose ω is tangent to AC at K, and let line ℓ intersect the circumcircle of $\triangle ADM$ again at E. Let the area of $\triangle ABC$ be S. Then, from the formula

$$S = \frac{1}{2}(a+b+c)\cdot r = \frac{1}{2}ab\sin C,$$

we have

$$\frac{a+b+c}{a} = \frac{b\sin C}{r} \Rightarrow \frac{(b+c-a)/2 + a}{a} = \frac{b}{(2r)/\sin C},$$

2 points.

which implies

$$\frac{AK+BC}{BC}=\frac{AC}{CD}\Rightarrow\frac{AK}{BC}=\frac{AD}{CD}.$$

2 points.

Noting that $\angle AEM=\angle CDI=\angle EDI=\angle EAM=90^\circ-\frac{C}{2}$, we conclude $ME=MA$. Applying Ptolemy's theorem to quadrilateral $ADME$ and noting that $AE=2AM\cdot\sin\frac{C}{2}$, we get

$$AM\cdot DE=EM\cdot AD+AE\cdot DM=AM\cdot AD+2AM\cdot\sin\frac{C}{2}\cdot DM.$$

3 points.

Thus,

$$DE=AD+2DM\cdot\sin\frac{C}{2}=AD+DI\cdot\sin\frac{C}{2}=AD+DK=AK.$$

1 point.

Therefore, $\frac{DE}{BC}=\frac{AD}{CD}$, which implies $\triangle AED\sim\triangle DBC$. Hence, $\angle AMD=\angle AED=\angle DBC$.

2 points.

Problem 4. Let $\mathcal{F}$ be a family of (distinct) subsets of the set $\{1, 2, \ldots, n\}$ such that for all $A, B \in \mathcal{F}$ we have that $A^c \cup B \in \mathcal{F}$, where A^c is the set of all members of $\{1, 2, \ldots, n\}$ that are not in A.

Prove that every $k \in \{1, 2, \ldots, n\}$ appears in at least half of the sets in $\mathcal{F}$.

(Stijn Cambie, Mohammad Javad Moghaddas Mehr)

First Solution. We start out by "cleaning up" our set family. We denote $[n] = \{1, 2, \ldots, n\}$, and refer to it as the *ground set*.

Firstly, if there exists a number $x \in \{1, 2, \ldots n\}$ which appears in every member of the family $\mathcal{F}$, remove it from all members of the family. Proving the claim of the problem for the remaining family clearly suffices, as x is in all sets of the family, and so in at least half of them.

0 points.

Additionally, while there exist two elements x, y such that for every $A \in \mathcal{F}$ we have

$$x \in A \iff y \in A$$

remove one of them from all the sets of the family, and do this until no such pairs remain. Proving the problem claim for the remaining family of sets clearly suffices, as every removed number has a corresponding number that is still in the ground set and appears in exactly as many sets of the family as the removed member originally did.

1 point.

Now, fix any pair of distinct elements $\{x, y\}$ of the ground set. We wish to show that there exist sets $A_{x,y}, A_{y,x} \in \mathcal{F}$ such that $x \in A_{x,y}, x \notin A_{y,x}$ and $y \in A_{y,x}, y \notin A_{x,y}$. As we ensured that x, y do not always appear together, one of them must exist. Assume without loss of generality that it is $A_{x,y}$.

Assuming that any set $A_{y,x} \in \mathcal{F}$ containing y but not x does not exist, this implies that for every $A \in \mathcal{F}$ we have

$$x \in A \implies y \in A.$$

Now, as y is not in every set of $\mathcal{F}$, there exists a set B such that $y \notin B$ and the previous implication implies that $x \notin B$. However, if we now consider the set $A_{x,y}^c \cup B \in \mathcal{F}$, it contains y but does not contain x, contradicting the nonexistence of a suitable $A_{y,x} \in F$.

2 points.

We now aim to show that for every $x \in [n]$, we have that $\{x\}^c \in \mathcal{F}$. Fix one such x, take some set $B \in \mathcal{F}$ such that $x \notin B$ and consider the set

$$B \cup \bigcup_{y \neq x} A_{x,y}^c.$$

This set contains all elements $y \neq x$, so it must be equal to $\{x\}^c$. It is a member of $\mathcal{F}$ by repeated $n-1$-fold application of the condition on members of $\mathcal{F}$.

4 points.

To finish, consider some $x \in [n]$ and some $B \in \mathcal{F}$ not containing x. We then have that $\{x\} \cup B \in \mathcal{F}$, so for every set in $\mathcal{F}$ that does not contain x we can find a unique one that does and we are done.

3 points.

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Second Solution. First we observe that for $A, B \in \mathcal{F}$ by using the rule on $A^c \cup B$ and B we get that

$$(A^c \cup B)^c \cup B = (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) = A \cup B \in \mathcal{F}. \tag{1}$$

2 points.

Now we take an arbitrary $x \in [n]$, let $\mathcal{A} = \{S \in \mathcal{F} : x \in S\}$ and $\mathcal{B} = \{S \in \mathcal{F} : x \notin S\}$. Then the set $T := \bigcup_{S \in \mathcal{B}} S$ is in $\mathcal{F}$ by using (1). Note that $x \notin T$ and that every $S \in \mathcal{B}$ is a subset of T.

3 points.

Now we'll prove that the function $f : \mathcal{B} \to \mathcal{A}$ defined by $f(S) = T^c \cup S$ is an injection.

3 points.

Suppose there exists $B_1, B_2 \in \mathcal{B}$ such that $T^c \cup B_1 = T^c \cup B_2$

$$\begin{aligned}
&\implies T \cap (T^c \cup B_1) = T \cap (T^c \cup B_2) \\
&\implies (T \cap T^c) \cup (T \cap B_1) = (T \cap T^c) \cup (T \cap B_2) \\
&\implies \emptyset \cup B_1 = \emptyset \cup B_2 \\
&\implies B_1 = B_2
\end{aligned}$$

where the third implication holds since $B_1, B_2 \subseteq T$. So f is injective $\implies |\mathcal{A}| \geqslant |\mathcal{B}|$.

2 points.

Third Solution. Obtain that for no two elements $x, y \in [n]$ holds $x \in A \iff y \in A$ for every $A \in \mathcal{F}$ as in the first solution.

1 point.

For arbitrary x, let T be the largest set not containing x. We claim $T = \{x\}^c$.
Assume the opposite, then every set $A \in \mathcal{F}$ containing x needs to contain T^c because otherwise $|A^c \cup T| > |T|$.

2 points.

Show that for any two sets $A, B \in \mathcal{F}$ their union $A \cup B \in \mathcal{F}$ is also in the family as shown in the Second Solution.

2 points.

If $A \in \mathcal{F}$ contains an element of T^c, then $|A \cup T^c| > |T|$ so A must contain x.

1 point.

These to combined imply that $x, y \in T^c$ belong to the exact same sets in $\mathcal{F}$ which is a contradiction with the claim at the beginning.

1 point.

Since we have $\{x\}^c \in \mathcal{F}$ we can finish the solution as in the First Solution.

3 points.