PROBLEMS AND SOLUTIONS OF THE 12$^{\text{TH}}$ EUROPEAN MATHEMATICAL CUP

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Senior Category

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Problem 1. Determine all sets of real numbers S such that:

- 1 is the smallest element of S,
- for all $x, y \in S$ such that $x > y$, $\sqrt{x^2 - y^2} \in S$.

(Adian Anibal Santos Sepčić)

Solution. All such sets are the set $\sqrt{\mathbb{N}} = \{\sqrt{n} \mid n \in \mathbb{N}\}$ and the sets $\sqrt{[n]} = \{\sqrt{k} \mid k \leqslant n, k \in \mathbb{N}\}$ for any $n \in \mathbb{N}$. It's easy to check that all such sets satisfy the problem's condition.

1 point.

We will now show that any S that satisfies the two conditions of the problem is of this form. It suffices to show that S can only contain square roots of positive integers and that $\sqrt{n} \in S$ implies that $\sqrt{m} \in S$ for any $m \leqslant n$.
First, note that we have $1 \in S$ so for any $x \in S$ with $x > 1$ we have $\sqrt{x^2 - 1} \in S$. Repeated application of this gives that $\sqrt{x^2 - n} \in S$ for any $n \in \mathbb{N}$ such that $n < x^2$.

3 points.

Now, assume that some number $x \in S$ is not the square root of an integer. We then immediately have $0 < x^2 - \lfloor x^2 \rfloor < 1$ as x^2 is not an integer, and the above consideration gives that $\sqrt{x^2 - \lfloor x^2 \rfloor} \in S$ and is strictly less than 1, which is a contradiction with $1 = \min(S)$. Therefore, we conclude that any element of S has to be the square root of an integer.

5 points.

Now, take some $\sqrt{n} \in S$. The above consideration immediately gives $\sqrt{n-1} \in S$ and we can conclude by induction that any smaller square root of an integer must also be a member of S, and we are done.

1 point.

Notes on marking:

- Any proof that shows that the members of S must necessarily be square roots of positive integers is worth 8 points. Of the other 2 points, one is assigned for correctly and completely writing the solution set, and one for noting that if $\sqrt{n} \in S$, then all smaller square roots $\sqrt{m}$ must also be in S. This must be explicitly commented on.

Problem 2. Let ABC be a triangle such that $\angle BAC = 90°$. The incircle of triangle ABC is tangent to the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F respectively. Let M be the midpoint of $\overline{EF}$. Let P be the projection of A onto BC and let K be the intersection of MP and AD. Prove that the circumcircles of triangles AFE and PDK have equal radius.

(Kyprianos-Iason Prodromidis)

First Solution. Let $S = EF \cap BC$, and let I be the center of the incircle of $\triangle ABC$ and let H be the orthocenter of $\triangle DEF$. Easy angle chase shows $\angle FDE = 45°$, and hence we have $DH = EF$ (both being equal to $\frac{\sqrt{2}}{2}$ times the radius of the incircle of $\triangle ABC$). Also, as $DIAH$ is a parallelogram, $AH \perp BC$ and $DH \perp EF$ which give AHP collinear. Our goal is to show $DKHP$ cyclic, because then diameters of the circles in question would be DH and EF respectively (latter because $HP \perp DP$), which we showed are equal.

3 points.

The main claim is the following: $IS \perp AD$ (This holds regardless of the right angle). One can prove it either via complex numbers (setting (DEF) as the unit circle) or by showing that AD is a polar of S with respect to the incircle of $\triangle ABC$. We will omit the proofs because the lemma is well-known.
Also, $AMPS$ is cyclic because of the right angles at M, P

2 points.

Now we have:

$$\angle HPK = \angle APM = \angle ASM = \angle ISM,$$

where the second equality follows from the $AMPS$ being cyclic, the third equality follows from $AEIF$ being a square, i.e. I is the reflection of A across EF.

2 points.

Using the lemma we have:

$$\angle ISM = \angle DAM = \angle KAM = \angle HDK,$$

where the first equality holds because $IS \perp AD$ and $SM \perp AM$, and the last equality follows from $DH \parallel AM$.

2 points.

These 2 chains of equality show $\angle HPK = \angle HDK$, which exactly means $HPDK$ is cyclic, and we conclude from the first paragraph.

1 point.

Second Solution. Let I be the circumcircle of DEF, notice that $AEFI$ is a square. Let BC, EF meet at S. It is clear that $AMKS$ is cyclic. Also A lies on the D-symmedian so since $IS \perp AD$, if IS, AD intersect at X, this point lies on the circumcircles of IEF, AMK.

3 points.

Let circles of DKL, AMK intersect at Y, it follows that $\angle DLK = \angle DYK$. But, $\angle DLK = \angle KTD + \angle MSA = 90° - \angle SDA + \angle MSA = \angle DSM = \angle MYK$. It follows that Y is on DM.

2 points.

Now, we shall show that the feet of E and F on DF and DE, respectively lie on the circle of DKL. By the theorem of radical axis, if circles FMY, EAF meet at Z, then Z would be on DF. We also have $\angle EZF = 90°$, since $\angle EAF = 90°$. Analogously, if the circles of EMY, EAF meet at W, we can generate a similar result. Thus, the circles of DKL, DZW would be the same, i.e., identical. If H is the orthocenter of DEF, we have $R_{DKL} = R_{DZW} = \frac{DH}{2} = \frac{IA}{2} = R_{AEF}$, as desired.

5 points.

Notes on marking:

- Citing the lemma as well-known won't cause point loss.
- In all incomplete computational solutions, only the geometric facts derived from the calculations will be worth points.

Problem 3. Let n be a positive integer. Let B_n be the set of all binary strings of length n. For a binary string $s_1 s_2 \dots s_n$, we define its twist in the following way. First, we count how many blocks of consecutive digits it has. Denote this number by b. Then, we replace s_b with $1 - s_b$. A string a is said to be a *descendant* of b if a can be obtained from b through a finite number of twists. A subset of B_n is called *divided* if no two of its members have a common descendant. Find the largest possible cardinality of a divided subset of B_n.

(Viktor Simjanoski)

Solution. For a string s, denote its twist by $f(s)$, and the number of blocks of consecutive digits it has by $b(s)$. Construct an undirected graph G on B_n with edges $(s, f(s))$ for all $s \in B_n$, and note that the largest possible cardinality of a divided subset of B_n is the number of connected components of the graph.

Each connected component of G contains exactly one cycle, and we aim to show that each cycle in the graph has a length of exactly 2.

1 point.

Assume that there exists a cycle $A \subseteq B_n$ which is not of length 2. First, fix some $s \in A$.

We wish to show that $1 < b(s) < n$. If we have $b(s) = 1$, then s is either the string with all ones or the string with all zeroes, and we can easily see that $f(s) = f^3(s)$ and $s \neq f(f(s))$, which contradicts $s \in A$. Similarly, if $b(s) = n$ then s is one of the two alternating strings and we arrive to the same conclusion.

1 point.

Now consider $x = s_{b(s)-1}, y = s_{b(s)}, z = s_{b(s)+1}$. A twist replaces y with $1 - y$, and depending on x, z changes $b(s)$ in the following ways:

- if $x = z \neq y$, we have $b(f(s)) = b(s) - 2$.
- if $x = y = z$, we have $b(f(s)) = b(s) + 2$.
- if $x \neq y = z$ or $x = y \neq z$, we have $b(f(s)) = b(s)$ and moreover we see that $f(f(s)) = s$ and the connected component of s has a cycle of length 2 so $s \notin A$.

We therefore see that if $s \in A$, we have $b(f(s)) = b(s) \pm 2$.

1 point.

Now, consider some $s \in A$. We then have $f^k(s) = s$ for some $k \in \mathbb{N}$. Take s such that $b(s) \geqslant b(f^m(s))$ for any $1 \leqslant m < k$, i.e. the element of the cycle with the largest number of blocks.

We have $b(f(s)) = b(s) - 2 = b(f^{-1}(s))$ by maximality and the previous proof. Notice that the application of f only changes the positions in s which are of the same parity as $b(s)$. We can see (as $b(s) = b(f^{-1}(s)) + 2$ that $s_{b(s)-3} = s_{b(s)-1} \neq s_{b(s)-2}$ and by similar reasoning $s_{b(s)-1} = s_{b(s)+1} \neq s_{b(s)}$.

1 point.

Now, consider the least $t > 0$ such that $b(f^t(s)) = b(s)$. By minimality of t, it follows that $f^t(s)_{b(s)} = 1 - s_{b(s)}$ but as we have $s_{b(s)+1} = 1 - s_{b(s)} = s_{b(s)-1}$ we obtain that $b(f(f^t(s))) = b(s) + 2$ which contradicts the maximality of $b(s)$ in A, so no such component A can exist.

4 points.

Now, let us count the possible cycles of length 2. Each cycle of length 2 occurs when we have $f(f(s)) = s$ and $b(s) = b(f(s))$, which gives $2 \leqslant b(s) \leqslant n - 1$.

We count by fixing either the left or right of the position $b(s) = k$ in a string s as one of $k - 1$ "break" points between 0/1 blocks in the string s and then counting that the other $k - 2$ block "breakpoints" can be assigned in $\binom{n-3}{k-2}$ ways to the remaining $n - 3$ spots between two symbols of s, with each assignment of blocks giving two distinct outcomes due to the choice of 0/1 in the starting block. This gives a total of

$$\sum_{k=2}^{n-1} 2\binom{n-3}{k-2} = 2^{n-2}$$

different cycles of length 2 and we are done.

2 points.

Problem 4. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for all positive integers x and y, the number $f(x) + y$ is a perfect square if and only if $x + f(y)$ is a perfect square. Prove that f is injective.

Remark. A function $f : \mathbb{N} \to \mathbb{N}$ is injective if for all pairs (x, y) of distinct positive integers, $f(x) \neq f(y)$ holds.

(Ivan Novak)

Solution. Suppose for the sake of contradiction that there exist positive integers a, b and c such that $f(a) = f(b) = c$ and $a > b$. Consider any $x > \sqrt{c}$.
Then, since $x^2 - c + c = x^2 - c + f(a) = x^2 - c + f(b)$ is a square, both $f(x^2 - c) + a$ and $f(x^2 - c) + b$ are squares. Since $a - b > 0$, we have the following bound:

$$a - b = (f(x^2 - c) + a) - (f(x^2 - c) + b) > \sqrt{f(x^2 - c) + a} + \sqrt{f(x^2 - c) + b}.$$

This implies that the function $x \mapsto f(x^2 - c)$ obtains only finitely many values since otherwise the bound wouldn't hold. By the pigeonhole principle, the expression $f(x^2 - c)$ obtains some fixed value m for infinitely many positive integers x.

1 point.

Consider a positive integer $y > \sqrt{m}$. Then $y^2 - m + f(x^2 - c) = y^2$ for infinitely many values of x. This implies that $f(y^2 - m) + x^2 - c$ is a square for infinitely many values of x. This implies $f(y^2 - m) - c = 0$, since it can be written as a difference of squares in infinitely many ways. Thus, $f(y^2 - m) = c$ for every $y > \sqrt{m}$.

1 point.

Now, $f(y^2 - m) = c$ for infinitely many y, so with the same argumentation as above we get $f(x^2 - c) = m$ for every $x > \sqrt{c}$.

Lemma. There exists a positive integer M such that for every positive integer z we have $f(z) \leqslant M$ or $f(z) \equiv m + 2 \pmod 4$

Proof. If $z = x^2 - c$ for some positive integer x then we have $f(z) = m$.

Now assume $z \neq x^2 - c$ for all positive integers x.
Let $y \in \mathbb{N}, y > \sqrt{m}$. If $f(z) + y^2 - m$ is a square, then $z + f(y^2 - m)$ is a square, but $f(y^2 - m) = c$, so this is contradiction with the choice of z.
So $f(z) + y^2 - m \neq x^2$ for all positive integers x.
From this, we have $f(z) \neq x^2 - y^2 + m$ for all positive integers x i y such that $y > \sqrt{m}$.

Let y_1 be the smallest positive integer such that $y_1 > \sqrt{m}$.
For every $y \geqslant y_1$ we have $f(z) \neq (y+1)^2 - y^2 + m = 2y + 1 + m$, so $f(z)$ is either smaller than $2y_1 + 1 + m$ or $f(z) - m \not\equiv 1 \pmod 2$.
For every $y \geqslant y_1$ we have $f(z) \neq (y+2)^2 - y^2 + m = 4y + 4 + m$, so $f(z)$ is either smaller than $4y_1 + 4 + m$ or $f(z) - m \not\equiv 0 \pmod 4$.
Now $M = 4y_1 + 4 + m$ satisfies the claim of the lemma.

□

5 points.

Take M which satisfies the lemma. Now take w such that $w+1, w+2, ..., w+M$ are not squares, and $w + m \equiv 0 \pmod 4$.
For some $d > \sqrt{f(w)}$
we have $d^2 - f(w) + f(w)$ is a square so $f(d^2 - f(w)) + w$ must be a square, but using the lemma on $z = d^2 - f(w)$ we get that $f(d^2 - f(w)) + w$ is either among $w + 1, w + 2, ..., w + M$ or congruent $2 + m + w \equiv 2$ modulo 4, so it cannot be a square.
Contradiction with the starting assumption, so f must be injective.

3 points.