Problems and solutions of the 12$^{\text{th}}$ European Mathematical Cup

9th December 2023 - 17th December 2023

Junior Category

Problem 1. Suppose a, b, c are positive integers such that

$$\gcd(a,b) + \gcd(a,c) + \gcd(b,c) = b + c + 2023.$$

Prove that $\gcd(b,c) = 2023$.

Remark: For positive integers x and y, $\gcd(x,y)$ denotes their greatest common divisor.

(Ivan Novak)

Solution. We want to prove $\gcd(a,b) = b$ and $\gcd(a,c) = c$, since then the equality from the problem statement implies $\gcd(b,c) = 2023$.

1 point.

Note that $\gcd(a,b)$ is a divisor of b and $\gcd(b,c)$ is a divisor of c, so we must have $\gcd(a,b) = \frac{b}{u}$ and $\gcd(a,c) = \frac{b}{v}$ for some positive integers u, v and we have

$$\frac{b}{u} + \frac{c}{v} + \gcd(b,c) = b + c + 2023.$$

We want to prove $u = v = 1$ so we need to eliminate other options. We do this by considering some cases.

1 point.

Case I. Both u and v are greater than 1.

In this case, we have $\gcd(a,b) \leqslant \frac{b}{2}$ and $\gcd(a,c) \leqslant \frac{c}{2}$, so we must have $\gcd(b,c) > \frac{b+c}{2}$. However, $\gcd(b,c)$ is a divisor of b and c so it is not greater than any of them and thus can't exceed their average, so we get a contradiction.

4 points.

Case II. $u = 1$ and $v > 1$.

In this case, we have

$$b + \frac{c}{v} + \gcd(b,c) = b + c + 2023,$$

which can be rewritten as

$$\gcd(b,c) = c + 2023 - \frac{c}{v}.$$

Since $c - \frac{c}{v} \geqslant \frac{c}{2}$, we have $\gcd(b,c) \geqslant \frac{c}{2} + 2023$, so it is a divisor of c greater than $\frac{c}{2}$. Thus, it must be equal to c, so we conclude that c divides b. However, note that $b \mid a$ since $\gcd(a,b) = b$, so we have $c \mid b \mid a$ which implies $c \mid a$. But then $\gcd(a,c) = c$, i.e. $v = 1$, a contradiction.

Case III. $u > 1$ and $v = 1$.

This case is analogous to Case II, so we don't need to consider it.

4 points.

Notes on marking:

- The points are all additive.

Problem 2. Let $n \geqslant 5$ be an integer. There are n points in the plane, no three of them collinear. Each day, Tom erases one of the points, until there are three points left. On the i-th day, for $1 \leqslant i \leqslant n-3$, before erasing that day's point, Tom writes down the positive integer $v(i)$ such that the convex hull of the points at that moment has $v(i)$ vertices. Finally, he writes down $v(n-2) = 3$. Find the greatest possible value that the expression

$$|v(1) - v(2)| + |v(2) - v(3)| + \ldots + |v(n-3) - v(n-2)|$$

can obtain among all possible initial configurations of n points and all possible Tom's moves.

Remark. A convex hull of a finite set of points in the plane is the smallest convex polygon containing all the points of the set (inside it or on the boundary).

(Ivan Novak, Namik Agić)

Solution. The answer is $2n-8$. The construction which achieves the bound is the following:

Take a semicircle Ω and mark $n-1$ points on it as $B_1, \ldots B_{n-1}$, in that order. Mark A as the intersection of tangents to Ω at B_1, B_{n-1} and consider A and B_i as n starting points. In the first move, erase A, and after the first move erase B_i in arbitrary order. It is easy to check that the first summand is $n-4$ and the remaining $n-4$ summands are 1, giving the desired bound.

3 points.

The proof of the bound is as follows:

The key idea is to look at the contributions of the individual vertices to the sum. We will prove that each vertex X has a contribution at most 2 to the sum. This is more or less immediate, the first possible contribution is when it becomes a vertex on a convex hull, and a second possible contribution is when it is erased from the hull (The sums after and before these 2 events are not affected by X, as well as between). Moreover, the points on an initial hull lose 1 possible contribution, and points of the final hull lose 1 possible contribution (consequence of the possibilities for contribution of a vertex).

3 points.

Let $z = v(1)$. From this we get an upper bound in contributions $2n - z - 3$ (Trivially, z is at least 3), Now we split into 2 cases to further sharpen this:

1 point.

- If there exists an index i such that $v(i) < v(i+1)$, let A be the erased vertex and let $B_1, \ldots B_k$ be the new vertices on the hull. The contributions from A and B_i for some i (WLOG say $i = 1$) cancel out, bringing the total bound of contributions to $2n - 3 - 3 - 2 = 2n - 8$.

2 points.

- If there is no such i, we get the bound of $n-3$ (as $v(i+1) \geqslant v(i) - 1$), and because $n \geqslant 5$, we also get an upper bound of $2n - 8 \geqslant n - 3$.

1 point.

In both cases we obtain the desired upper bound and the proof is complete.

Notes on marking:

- If a solution does not discuss the case where $v(i)$ is strictly decreasing (i.e. misses the discussion as in second bullet point), **1 point** should be deducted.

Problem 3. Consider an acute-angled triangle ABC with $|AB| < |AC|$. Let M and N be the midpoints of segments $\overline{BC}$ and $\overline{AB}$, respectively. The circle with diameter $\overline{AB}$ intersects the lines BC, AM and AC at D, E, and F, respectively. Let G be the midpoint of $\overline{FC}$. Prove that the lines NF, DE and GM are concurrent.

(Michal Pecho)

Solution. Let A' be the reflection of A across M, and let F' be the reflection of F across N. Easy angle chase gives that D and F are feet of altitudes from A, B respectively.

1 point.

First, $\angle DCA' = \angle ABC = \angle DEA'$, meaning that $DECA'$ is cyclic. As $NF = NA$, $FB \parallel AC \parallel BA'$, which gives F, B, A' collinear. Now we have $\angle F'FA = \angle BAC = \angle CA'B = \angle CA'F$, which yields $F'FCA'$ cyclic. $=1$

3+3 point.

From the radical center theorem on $(BDEFA), (DECA'), (F'FCA')$ we know that CA', FF', DE are concurrent. Let Z be the point of concurrency.

2 points.

As $\angle F'FC = \angle FCA'$,we get that $\angle ZFC = \angle ZCF$, i.e. Z lies on the perpendicular bisector of FC. But this bisector is exactly GM (as $MF = MC$ and $GF = GC$). Finally, Z also lies on GM and the three lines from the statement are indeed concurrent at Z

1 point.

Notes on marking:

- There is a projective approach to the problem, sketched in a marking scheme. Other computational methods seem hard to execute. For partial non-synthetic solutions, only the parts which are geometric interpretations will be awarded points.
- If the solution marks the additional points as in the above, points will be given only for obtaining useful results, not solely on marking them.

Problem 4. We say a 2023-tuple of nonnegative integers $(a_1, a_2, \dots a_{2023})$ is *sweet* if the following conditions hold:

- $a_1 + a_2 + \ldots + a_{2023} = 2023$,
- $\frac{a_1}{2^1} + \frac{a_2}{2^2} + \ldots + \frac{a_{2023}}{2^{2023}} \leqslant 1$.

Determine the greatest positive integer L such that

$$a_1 + 2a_2 + \ldots + 2023a_{2023} \geqslant L$$

holds for every sweet 2023-tuple $(a_1, a_2, \dots, a_{2023})$.

(Ivan Novak)

Solution. Let $a_1, a_2, \dots$ be a sweet sequence for which the least value L of $a_1 + 2a_2 + 3a_3 + \dots$ is achieved.
Suppose that there are two nonconsecutive indices $i < k$ with $a_i, a_k > 0$.
Consider the sequence

$$(a_1, a_2, \dots, a_{i-1}, a_i - 1, a_{i+1} + 2, a_{i+2}, \dots, a_{k-1}, a_k - 1, a_{k+1}, \dots),$$

i.e. the sequence in which the i-th and k-th term are reduced by 1 and the $i+1$-th term is increased by 2.
We claim that this sequence is also sweet and that it achieves the value not greater than L.
The sum of its elements is unchanged so the first condition is satisfied.
For the second condition, note that

$$\frac{a_i - 1}{2^i} + \frac{a_{i+1} + 2}{2^{i+1}} + \frac{a_k - 1}{2^k} = \frac{a_i}{2^i} + \frac{a_{i+1}}{2^{i+1}} + \frac{a_k}{2^k} - \frac{1}{2^k},$$

so the sum in the second condition decreases, so it remains smaller than 1.
Finally, we claim that the value of the sum we're minimising didn't increase. Indeed, we have

$$i(a_i - 1) + (i+1)(a_{i+1} + 2) + k(a_k - 1) = ia_i + (i+1)a_{i+1} + ka_k - (k - i - 2),$$

and since $k \geqslant i + 2$, this means the sum didn't increase.
Repeating this transformation finitely many times, we obtain a sequence which obtains the minimum and which doesn't have two nonconsecutive indices $i < k$ with $a_i, a_k > 0$. Thus, it suffices to check sequences of the form

$$(0, 0, \dots, 0, a, b, 0, 0, \dots),$$

with $i - 1$ leading zeroes for some positive integer i, and with $a > 0$, $b \geqslant 0$.

6 points.

In this case, we have the conditions

$$a + b = 2023,$$
$$2a + b \leqslant 2^{i+1},$$

and we're minimising the expression

$$f(a, b, i) = ia + (i+1)b = 2023i + b.$$

Since $0 \leqslant b < 2023$, the optimal value of i is the least one for which the two conditions can be satisfied. We must have $2^{i+1} = 2a + b > 2023$, which holds if and only if $i + 1 \geqslant 11$, i.e. we must check $i = 10$.

3 points.

When $i = 10$, we have $a + b = 2023$, $2a + b \leqslant 2048$, which gives $b \geqslant 1998$, or

$$10a + 11b \geqslant 20230 + 1998 = 22228.$$

This value can be obtained for $a = 25$ and $b = 1998$, so we conclude that $L = 22228$.

1 point.

Notes on marking:

- In the first part worth 6 points, 2 points will be awarded if the solution states that we can WLOG have equality in (2).
- Failed attempts of smoothing the sequence will be worth points depending on how close is it to the correct one (1 or 2 points). If it also contains first remark, the total sum is 1+pts on smoothing (not completely additive with smoothing)