12^{TH} European Mathematical Cup

9^{th} December 2023 - 17^{th} December 2023

Junior Category

Problem 1. Suppose a, b, c are positive integers such that

$$\gcd(a, b) + \gcd(a, c) + \gcd(b, c) = b + c + 2023.$$

Prove that $\gcd(b, c) = 2023$.

Remark: For positive integers x and y, $\gcd(x, y)$ denotes their greatest common divisor.

(Ivan Novak)

Problem 2. Let $n \geqslant 5$ be an integer. There are n points in the plane, no three of them collinear. Each day, Tom erases one of the points, until there are three points left. On the i-th day, for $1 \leqslant i \leqslant n-3$, before erasing that day's point, Tom writes down the positive integer $v(i)$ such that the convex hull of the points at that moment has $v(i)$ vertices. Finally, he writes down $v(n-2) = 3$. Find the greatest possible value that the expression

$$|v(1) - v(2)| + |v(2) - v(3)| + \ldots + |v(n-3) - v(n-2)|$$

can obtain among all possible initial configurations of n points and all possible Tom's moves.

Remark. A convex hull of a finite set of points in the plane is the smallest convex polygon containing all the points of the set (inside it or on the boundary).

(Ivan Novak, Namik Agić)

Problem 3. Consider an acute-angled triangle ABC with $|AB| < |AC|$. Let M and N be the midpoints of segments $\overline{BC}$ and $\overline{AB}$, respectively. The circle with diameter $\overline{AB}$ intersects the lines BC, AM and AC at D, E, and F, respectively. Let G be the midpoint of $\overline{FC}$. Prove that the lines NF, DE and GM are concurrent.

(Michal Pecho)

Problem 4. We say a 2023-tuple of nonnegative integers $(a_1, a_2, \ldots a_{2023})$ is *sweet* if the following conditions hold:

- $a_1 + a_2 + \ldots + a_{2023} = 2023$,
- $\frac{a_1}{2^1} + \frac{a_2}{2^2} + \ldots + \frac{a_{2023}}{2^{2023}} \leqslant 1$.

Determine the greatest positive integer L such that

$$a_1 + 2a_2 + \ldots + 2023a_{2023} \geqslant L$$

holds for every sweet 2023-tuple $(a_1, a_2, \ldots, a_{2023})$.

(Ivan Novak)

Time: 240 minutes. *Each problem is worth 10 points.*

The use of calculators or any other instruments except rulers and compasses is not permitted. Cheating in any form is strictly prohibited. Sharing or posting the problems or solutions before the end of the competition (18th of December) to anyone who might participate is strictly prohibited.