11^{TH} European Mathematical Cup

10^{th} December 2022 - 18^{th} December 2022

Junior Category

Problems and Solutions

Problem 1. Find all positive integers n for which there exist three positive divisors a, b, c of n such that $a > b > c$ and

$$a^2 - b^2,\ \ b^2 - c^2,\ \ a^2 - c^2$$

are also divisors of n.

(Kims Georgs Pavlovs)

Solution. The answer is all positive integers n which are divisible by 60.

Suppose that n is divisible by 60. Let $a = 4$, $b = 2$, $c = 1$. Then $a^2 - b^2 = 12$, $b^2 - c^2 = 3$, $a^2 - c^2 = 15$. All six of those numbers are divisors of 60, so they're also divisors of n, so any n which is a multiple of 60 is indeed a solution.

2 points.

Now suppose that n is a number for which such a, b, c exist.

Some two numbers among a, b, c have the same parity. Without loss of generality let a and b be such numbers. Then $a - b$ and $a + b$ are both even, so 4 is a divisor of $a^2 - b^2$, so 4 also divides n. We conclude that n is divisible by 4.

2 points.

If any of a, b, c is divisible by 3, then n is also divisible by 3. Otherwise, two numbers among a, b, c give the same remainder upon division by 3. Without loss of generality let a and b be such numbers. Then 3 is a divisor of $a^2 - b^2$, so 3 also divides n. We conclude that in any case, n is divisible by 3.

2 points.

If any of a, b, c is divisible by 5, then n is also divisible by 5. Otherwise, none of a, b, c is divisible by 5. However, a square of a number which isn't divisible by 5 can only give remainder 1 or 4 upon division by 5.

Thus, some two numbers among a^2, b^2, c^2 have the same remainder upon division by 5, so their difference is divisible by 5. We conclude that in any case, n is divisible by 5.

3 points.

Since $3, 4$ and 5 all divide n, then their least common multiple, which is 60, also divides n.

1 point.

Notes on marking:

- If a contestant proves that for some $a > 1$, all numbers of the form $60 \cdot a \cdot k$ for $k \in \mathbb{N}$ are solutions, they should be awarded **1 point** out of possible **2 points** for the first part of the solution.
- If a contestant proves that any n which is a solution is even, they should be awarded **1 point** out of possible **2 points** given for proving divisibility by 4.

Problem 2. Find all pairs of positive real numbers (x, y) such that xy is an integer and

$$x + y = \lfloor x^2 - y^2 \rfloor.$$

(Ivan Novak)

First Solution. Let (x, y) be a pair satisfying the problem's condition.
Note that $x + y$ and xy are both positive integers, so $(x - y)^2 = (x + y)^2 - 4xy$ is also a positive integer.

4 points.

Let $D = (x - y)^2$, and let $a = x + y$. We have

$$a = \lfloor a\sqrt{D} \rfloor,$$

where a and D are both positive integers. Furthermore, note that $D = (x + y)^2 - 4xy$ gives remainder 0 or 1 upon division by 4. If $D > 1$, then $D \geqslant 4$, and $\lfloor a\sqrt{D} \rfloor \geqslant \lfloor 2a \rfloor = 2a > a$, which is a contradiction. Thus, $D = 1$.

3 points.

This means that $x - y = 1$. Since $x + y = a$, we must have

$$x = \frac{a+1}{2},$$
$$y = \frac{a-1}{2}$$

for some positive integer a. Since $y > 0$, we must have $a > 1$. Since xy is an integer, $\frac{a^2-1}{4}$ must be an integer. Thus, a is odd. Let $a = 2n + 1$ for some positive integer n. Then

$$(x, y) = (n + 1, n).$$

It's easy to check that all such pairs satisfy the problem's conditions.

3 points.

Second Solution. Note that $x > y$ since $x^2 - y^2 \geqslant x + y > 0$.
Let $a = x + y$ and $b = xy$. We then have $x^2 - ax + b = 0$ and $y^2 - ay + b = 0$. This means that x and y are the roots of the polynomial $t^2 - at + b$, so, since $x > y$, we have

$$x = \frac{a + \sqrt{D}}{2},$$
$$y = \frac{a - \sqrt{D}}{2},$$

where $D = a^2 - 4b$ is a positive integer.

2 points.

Direct calculation yields $x^2 - y^2 = a\sqrt{D}$.

2 points.

Thus, we again obtain the equality

$$a = \lfloor a\sqrt{D} \rfloor.$$

The rest of the solution is the same as in the First Solution.

6 points.

Third Solution. We let $a = x+y$ (note it must be a positive integer) and then the equality from the statement implies that $a \leqslant x^2 - y^2 < a + 1$ which upon division by a implies

$$1 \leqslant x - y < 1 + \frac{1}{a}.$$

1 point.

Adding $a = x + y$ to both inequalities implies

$$a + 1 \leqslant 2x < a + 1 + \frac{1}{a} \implies \frac{a+1}{2} \leqslant x < \frac{a+1}{2} + \frac{1}{2a}.$$

2 points.

Now, write $x = \frac{a+1}{2} + \varepsilon$ for some ε. We see that $y = \frac{a-1}{2} - \varepsilon$ and also that $0 \leqslant \varepsilon < \frac{1}{2a} \leqslant \frac{1}{2}$.
We can compute

$$xy = \frac{a^2-1}{4} - (\varepsilon^2 + \varepsilon) \in \mathbb{N}$$

and note that $0 \leqslant \varepsilon^2 + \varepsilon < \frac{3}{4}$.

2 points.

If a is even, $a^2 - 1$ gives remainder 3 upon division by 4 so the fractional part of $\frac{a^2-1}{4}$ is $\frac{3}{4}$ and due to $\varepsilon^2 + \varepsilon < \frac{3}{4}$ we have that xy is not an integer.

2 points.

If a is odd, $a^2 - 1$ is divisible by 4 so we must have $\varepsilon^2 + \varepsilon = 0$ and $\varepsilon = 0$ and we obtain the solutions

$$(x, y) = \left(\frac{a+1}{2}, \frac{a-1}{2}\right) = (n+1, n)$$

for any odd integer a, that is, any positive integer n.

3 points.

Fourth Solution. Again, denote $a = x + y$ and $D = (x-y)^2$ (both are integers as in the First Solution or Second Solution). Thus we again have

$$a = \lfloor a\sqrt{D} \rfloor.$$

4 points.

As in the Third Solution, we have that $a = \lfloor a\sqrt{D} \rfloor$ implies that

$$1 \leqslant x - y = \sqrt{D} < 1 + \frac{1}{a}$$

.

1 point.

Applying the inequality of arithmetic and geometric means gives

$$a = x + y \geqslant 2\sqrt{xy} \geqslant 2$$

but as $x \neq y$ equality can not hold. This gives that $a \geqslant 3$ so $\sqrt{D} \leqslant \frac{4}{3}$ and as D is an integer, $D = 1$.

2 points.

The rest of the solution is the same as in the First Solution.

3 points.

Notes on marking:

- In all four solutions, in the final part (last **3 points**), minor flaws should result in a **1 point** deduction.
- If a contestant has not made the steps prior to the final part of the solution, they can score at most **2 points** for the last part.
- Stating that all pairs of the form $(n+1, n)$ are a solution is worth at most **1 point** out of the last **3 points**.
- Points from different solutions are not additive.

Problem 3. Let ABC be an acute-angled triangle with $|BC| < |AC|$. Let I be the incenter and τ the incircle of ABC, which touches BC and AC at points D and E, respectively. The point M is on τ such that BM is parallel to DE and M and B are on the same side of the angle bisector of $\angle BCA$. Let F and H be the intersections of τ with BM and CM different from M, respectively. Let J be a point on the line AC such that JM is parallel to EH. Let K be the intersection of JF and τ different from F.

Prove that the lines ME and KH are parallel.

(Steve Vo Dinh)

First Solution. Let T be the intersection of CF and τ different from F. Consider the homotethy centered at C that sends H to M.
It also sends T to F because of symmetry around the bisector of angle $\angle ACB$.

1 point.

Furthermore, it sends E to J because JM is parallel to EH.

1 point.

Since CE is tangent to τ, CJ is tangent to the circumcircle of $\triangle MFJ$ using the aforementioned homotethy.

4 points.

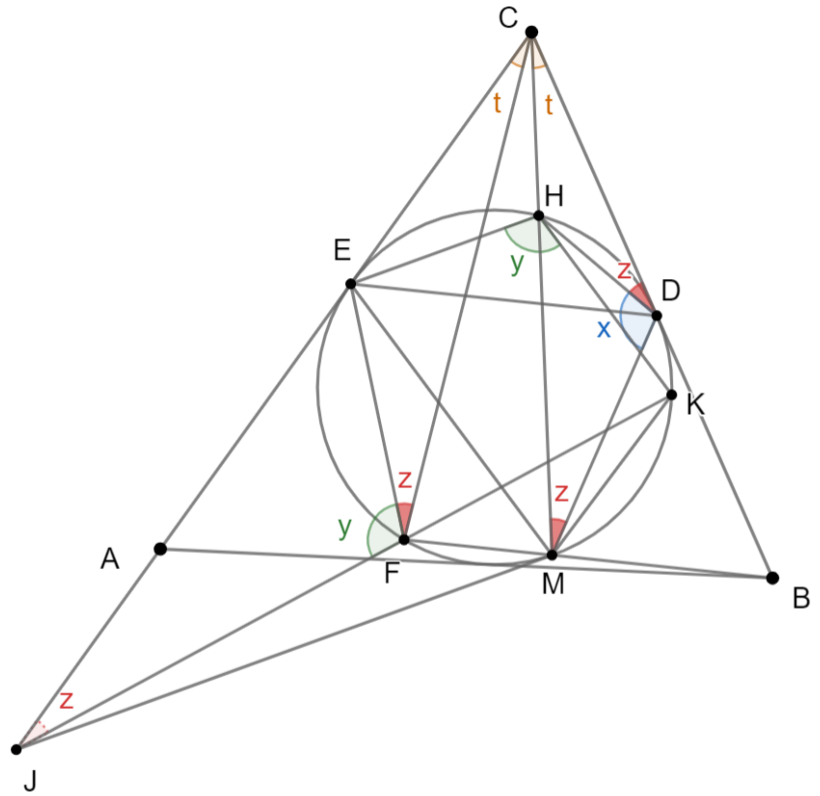
By the tangent chord lemma we have $\angle CJM = \pi - \angle MFJ$.

1 point.

Now notice that $\angle EMH = \angle CEH = \angle CJM = \pi - \angle MFJ = \angle KFM = \angle MHK$.
Thus, ME and KH are parallel, as desired.

3 points.

Second Solution. Let t denote the angle $\angle JCF$, and let z denote the angle $\angle EFC$.



We claim that $\triangle CJF \sim \triangle CDH$.
We first note that $\angle JCF = \angle DCM = t$, $CD = CE$ and $CM = CF$ using the symmetry around the bisector of angle $\angle ACB$. Namely, let ℓ be the mentioned bisector. Then since FM and ED are parallel, we conclude that $EDFM$ is an isosceles trapezoid, so D and E are symmetric around ℓ, and the same holds for F and M.

1 point.

Using the fact that $JM \parallel EH$, the triangles JMC and EHC are similar, and we get $\frac{CJ}{CM} = \frac{CE}{CH}$, or equivalently $\frac{CJ}{CF} = \frac{CD}{CH}$

1 point.

. We are now done by the side-angle-side theorem.

4 points.

Now, notice that from the tangent chord lemma applied to BC and HD, we have $\angle HDC = \angle HMD$. From this, it follows $\angle EFC = \angle CMD = \angle HMD$ because of symmetry around angle $\angle ACB$.

Since we also have $\angle HDC = \angle CJF$ from the aforementioned similarity, we obtain $\angle HDC = \angle CJF = \angle EFC = z$.

1 point.

From triangle JFC, we have $\angle KHE = \angle JFE = \pi - \angle CJF - \angle JCF - \angle EFC = \pi - 2z - t$.

From triangle CDM we have $\angle MDH = \pi - \angle DMH - \angle DCH - \angle HDC = \pi - 2z - t$. Thus, $\angle MDH = \angle EHD$, which implies that $ED = HM$ as the angles across those chords are the same. Thus, $EMKH$ is an isosceles trapezoid, so ME and KH are parallel, as desired.

3 points.

Notes on marking:

- In both solutions, contestants that have done the angle-chase (final part of the solution), but haven't done the previous steps (up to minor flaws), are awarded **1 point** out of **3 points** for the final part, since it can be helpful, but it isn't very useful without the previous steps.
- Points from different solutions are not additive.

Problem 4. Let $X = \{1, 2, 3, \ldots, 300\}$. A collection F of distinct (not necessarily non-empty) subsets of X is *lovely* if for any three (not necessarily distinct) sets A, B, C in F, at most three out of the following eight sets are non-empty:

$$A \cap B \cap C, \quad \overline{A} \cap B \cap C, \quad A \cap \overline{B} \cap C, \quad A \cap B \cap \overline{C},$$
$$\overline{A} \cap \overline{B} \cap C, \quad \overline{A} \cap B \cap \overline{C}, \quad A \cap \overline{B} \cap \overline{C}, \quad \overline{A} \cap \overline{B} \cap \overline{C},$$

where $\overline{S}$ denotes the set of all elements of X that are not in S.

What is the greatest possible number of sets in a lovely collection?

(Miroslav Marinov)

First Solution. We claim that $|F| \leqslant 8$.

If we apply the condition to the triple (A, B, B) for some distinct sets $A, B \in F$ we obtain that at most three of the sets $A \cap B, B \setminus A, A \setminus B, X \setminus (A \cup B)$ are nonempty. Therefore, for each pair of distinct sets $A, B \in F$ we either have that $A \subset B$ or $B \subset A$ or A, B disjoint or $A \cup B = X$.

1 point.

Now fix an F of maximal size and assume $|F| \geqslant 9$. Consider $G = F \setminus \{\emptyset, X\}$ which has $|G| \geqslant 7$. We wish to show there exist $A, B \in G$ with $A \cup B = X$ and nonempty intersection. Assume the opposite, that all A, B in G are either disjoint or contained in each other.

First, assume a pair of sets with $A \supset B$ exists in G.
If there exists $C \in G$ such that $A \supset B \supset C$ we obtain that

$$\begin{aligned} C &= A \cap B \cap C \\ B \setminus C &= A \cap B \cap \overline{C} \\ A \setminus B &= A \cap \overline{B} \cap \overline{C} \\ X \setminus A &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

are all nonempty, a contradiction. Similarly, A has no superset in G.

1 point.

If there exist distinct $C, D \in G$ such that $A \supset B, C, D$, the previous consideration forces B, C, D to be disjoint and we obtain that

$$\begin{aligned} C &= A \cap \overline{B} \cap C \\ B &= A \cap B \cap \overline{C} \\ X \setminus A &= \overline{A} \cap \overline{B} \cap \overline{C} \\ A \setminus (B \cup C) &= A \cap \overline{B} \cap \overline{C} \end{aligned}$$

and as the first three sets are clearly nonempty, the last one must be empty and we have $B \cup C = A$. However, we now similarly get $D \cup B = D \cup C = A$ and D disjoint from B, C so $B = C$, a contradiction, so A has at most two subsets in G.

1 point.

Now, due to $|G| \geqslant 7$ we can choose distinct $C, D \in G$ with C, D both disjoint from A. We now have the sets

$$\begin{aligned} C &= \overline{A} \cap \overline{B} \cap C \\ B &= A \cap B \cap \overline{C} \\ A \setminus B &= A \cap \overline{B} \cap \overline{C} \\ X \setminus (A \cup C) &= \overline{A} \cap \overline{B} \cap \overline{C} \end{aligned}$$

and as the first three sets are clearly nonempty, the last one must be empty so $C = \overline{A}$. However, now we have a similar consideration for D so $C = D = \overline{A}$, a contradiction. Therefore, any pair of sets $A, B \in F$ are disjoint.

2 points.

Now, take some $A, B, C, D \in F$ and note that

$$A = A \cap \overline{B} \cap \overline{C}$$
$$B = \overline{A} \cap B \cap \overline{C}$$
$$C = \overline{A} \cap \overline{B} \cap C$$
$$X \setminus (A \cup B \cup C) = \overline{A} \cap \overline{B} \cap \overline{C}$$

so as the first three are clearly nonempty, the last one must be empty so we have $A \cup B \cup C = X$ and specially $C = \overline{A \cup B}$. However, the same consideration now applies to D so we obtain $D = C$, a contradiction.

1 point.

Finally, take a pair $A, B \in G$ such that $A \cup B = X$ and $A \cap B \neq \emptyset$. Consider some other $C \in G$. As all three sets $A \cap B$, $A \cap \overline{B}$, $\overline{A} \cap B$ are nonempty, each has to have a nonempty intersection either with C or with $\overline{C}$. As we can have at most three such nonempty intersections, we have that all of them are either a subset of C or a subset of $\overline{C}$. As those three sets partition X, this gives us at most $2^3 = 8$ possible choices for C and as $C \in G = F \setminus \{\emptyset, X\}$ and $C \neq A, C \neq B$, we cannot have that they're all contained in C or in $\overline{C}$ and we cannot have $C = A$ or $C = B$ so there are at most 4 possibilities for C and we obtain $|G| \leqslant 6$, a contradiction.

3 points.

Finally, we have that $|G| \leqslant 6$ so $|F| \leqslant 8$ as desired.

One example of this is F containing $\emptyset, \{1, 2, \ldots, 100\}, \{101, 102, \ldots 200\}, \{201, 300\}$ and their complements. It's easy to check that this is a lovely family.

1 point.

Second Solution. Let F be a lovely family with the maximum possible number of sets. We claim that F contains less than 9 sets. Suppose for the sake of contradiction that it contains more than 8 sets.

Note that the problem's condition is symmetric under taking complements. In other words, the sets A, B, C satisfy the problem's condition if and only if the sets $\overline{A}, B, C$ satisfy it. Because of this, we may assume that for any $A \in F$ we also have $\overline{A} \in F$. Since F consists of least 9 sets, it follows that F contains at least 5 sets which all contain the number 1.

1 point.

We'll prove this is impossible.

We rephrase the problem. Represent each set $A \in F$ with a 300×1 column consisting of zeroes and ones, so that the kth number in the column is 0 if $k \notin A$ and 1 if $k \in A$.

Let A, B, C, D, E be five sets from F which all contain 1. Consider a 300×5 table consisting of 5 columns corresponding to A, B, C, D, E respectively.

Now note that the problem's condition rephrases as follows: for any choice of three columns from the table, the 300×3 table consisting of those three columns contains at most 3 different rows out of possible 8.

1 point.

Lemma. Suppose that there is a row r in the mentioned 300×5 table which doesn't contain all ones and contains at least two ones. Then for any two columns which contain a one in the row r, one of them contains all ones.

Proof. Without loss of generality we may assume that the row r is of the form $110xy$, where $x, y \in \{0, 1\}$. Since the first two columns can't be the same, there must exist a row whose first two entries are distinct. Without loss of generality, this row is of the form $10uvw$ where $u, v, w \in \{0, 1\}$. Now look at the first three columns in the table. They contain rows 111, $10u$, 110. Those are three distinct rows, so the first three entries of any row in the table are 111, $10u$ or 110. This implies that every row must start with a one, which means that the first column contains all ones.

□

3 points.

From this lemma, it immediately follows that there is no row which contains more than 2 and less than 5 ones, and if there is a row which contains exactly 2 ones, then there is a column which contains all ones.

In any case, take a look at the four columns which do not contain all ones. Each of the rows in the corresponding 300×4 table either contains no ones, a single one, or four ones.

Since the columns need to be distinct, there must exist at least three out of four of the following rows:

1000

0100

0010

0001

2 points.

Without loss of generality we can assume that the table contains the first three among them. The table also contains the first row with all ones. However, looking at the first three columns then yields a contradiction, since there exist the following four distinct rows:

111

100

010

001

2 points.

The example which proves that a lovely family can contain 8 sets is the same as in the first solution.

1 point.

Notes on marking:

- In the First Solution, solving the case when there exist A, B whose union is X and intersection is nonempty is worth **3 points**. It is possible to score **1 point** or **2 points** on this part of the problem if a contestant makes partial progress or has a minor flaw in their proof of this case.