10TH European Mathematical Cup

11th December 2021 - 19th December 2021

Senior Category

Mladi nadareni matematičari **Marin Getaldić**

Problem 1. Alice drew a regular 2021-gon in the plane. Bob then labelled each vertex of the 2021-gon with a real number, in such a way that the labels of consecutive vertices differ by at most 1. Then, for every pair of non-consecutive vertices whose labels differ by at most 1, Alice drew a diagonal connecting them. Let d be the number of diagonals Alice drew. Find the least possible value that d can obtain.

(Ivan Novak)

Problem 2. Let ABC be a triangle and let D, E and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Let $X \neq A$ be the intersection of AD with the circumcircle of ABC. Let Ω be the circle through D and X, tangent to the circumcircle of ABC. Let Y and Z be the intersections of the tangent to Ω at D with the perpendicular bisectors of segments $\overline{DE}$ and $\overline{DF}$, respectively. Let P be the intersection of YE and ZF and let G be the centroid of ABC. Show that the tangents at B and C to the circumcircle of ABC and the line PG are concurrent.

(Jakob Jurij Snoj)

Problem 3. Let $\mathbb{N}$ denote the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$x^2 - y^2 + 2y(f(x) + f(y))$$

is a square of an integer for all positive integers x and y.

(Ivan Novak)

Problem 4. Find all positive integers d for which there exist polynomials $P(x)$ and $Q(x)$ with real coefficients such that degree of P equals d and

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2.$$

(Ivan Novak)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.