10$^{\text{TH}}$ European Mathematical Cup

11th December 2021 - 19th December 2021

Junior Category

MLADI NADARENI MATEMATIČARI
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Problem 1. We say that a quadruple of nonnegative real numbers (a, b, c, d) is *balanced* if

$$a + b + c + d = a^2 + b^2 + c^2 + d^2.$$

Find all positive real numbers x such that

$$(x - a)(x - b)(x - c)(x - d) \geqslant 0$$

for every balanced quadruple (a, b, c, d).

(Ivan Novak)

Problem 2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Let X and Y be points on the minor arc $\widehat{BC}$ of the circumcircle of ABC such that $|BX| = |XY| = |YC|$. Suppose that there exists a point N on the segment $\overline{AY}$ such that $|AB| = |AN| = |NC|$. Prove that the line NC passes through the midpoint of the segment $\overline{AX}$.

(Ivan Novak)

Problem 3. Let ℓ be a positive integer. We say that a positive integer k is *nice* if $k! + \ell$ is a square of an integer. Prove that for every positive integer $n \geqslant \ell$, the set $\{1, 2, \ldots, n^2\}$ contains at most $n^2 - n + \ell$ nice integers.

(Théo Lenoir)

Problem 4. Let n be a positive integer. Morgane has coloured the integers $1, 2, \ldots, n$. Each of them is coloured in exactly one colour. It turned out that for all positive integers a and b such that $a < b$ and $a + b \leqslant n$, at least two of the integers among a, b and $a + b$ are of the same colour. Prove that there exists a colour that has been used for at least $2n/5$ integers.

(Vincent Jugé)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.