8^{TH} EUROPEAN MATHEMATICAL CUP

14^{th} December 2019 - 22^{th} December 2019

Senior Category

Problems and Solutions

Problem 1. For positive integers a and b, let $M(a, b)$ denote their greatest common divisor. Determine all pairs of positive integers (m, n) such that for any two positive integers x and y such that $x \mid m$ and $y \mid n$,

$$M(x + y, mn) > 1.$$

(Ivan Novak)

First Solution. We will prove that there are no solutions. Let m and n be any positive integers.

Let P denote the product of all primes which divide n and don't divide m. Then m is a divisor of m and P is a divisor of n, but we'll prove that $m + P$ and mn are relatively prime.

4 points.

Let p be any prime divisor of mn.
If p divides m, then p doesn't divide P and therefore p doesn't divide $m + P$.

3 points.

If p doesn't divide m, then p divides n, and then p divides P by definition of P, which implies that p doesn't divide $m + P$.

3 points.

Hence, $m + P$ and mn have no common prime factors, which implies they are relatively prime. Hence, there are no solutions.

Second Solution. We will prove that there are no solutions. Assume for the sake of contradiction that (m, n) was a solution. We will recursively construct an infinite unbounded sequence of pairs of positive integers $(x_k, y_k)_{k\in\mathbb{N}}$ such that $x_k \mid m$, $y_k \mid n$ and $M(x_k, y_k) = 1$.

1 point.

Then either $(x_k)_{k\in\mathbb{N}}$ or $(y_k)_{k\in\mathbb{N}}$ will be unbounded, but $x_k \leqslant m$ and $y_k \leqslant n$ for all $k \in \mathbb{N}$, which will yield a contradiction.

1 point.

Let $(x_1, y_1) = (1, 1)$. Let $k \in \mathbb{N}$. Suppose we have constructed (x_k, y_k) satisfying all of the above conditions. Then since (m, n) is a solution, there exists a prime divisor p of both mn and $x_k + y_k$.

1 point.

If p divides m, then let $(x_{k+1}, y_{k+1}) = (px_k, y_k)$.

2 points.

If p divides n and doesn't divide m, let $(x_{k+1}, y_{k+1}) = (x_k, py_k)$.

2 points.

In both cases x_{k+1} divides m and y_{k+1} divides n.

1 point.

Also, $M(x_{k+1}, y_{k+1}) = 1$ because p does not divide neither x_k nor y_k (as x_k and y_k are relatively prime and p divides $x_k + y_k$). Hence, the construction is valid.

2 points.

Notes on marking:

- In the First solution, there are different choices for pairs of divisors whose sum is relatively prime with mn. For example, one can take $(\text{rad}(m), \dfrac{\text{rad}(mn)}{\text{rad}(m)})$, where $\text{rad}(x)$ denotes the product of all prime divisors of x. If a student finds such a pair and claims that it is a solution without proving that their sum is relatively prime with mn, and if the proof is as straightforward as in the official solution, he should still get **4 points** from the first part of the solution.

Problem 2. Let n be a positive integer. An $n \times n$ board consisting of n^2 cells, each being a unit square coloured either black or white, is called *convex* if for every black coloured cell, both the cell directly to the left of it and the cell directly above it are also coloured black. We define the *beauty* of a board as the number of pairs of its cells (u, v) such that u is black, v is white and u and v are in the same row or column. Determine the maximum possible beauty of a convex $n \times n$ board.

(Ivan Novak)

First Solution. We colour the board so that in the i-th row, the leftmost $n+1-i$ cells are black. We'll call this board *the Unicorn*.

1 point.

The beauty of this board equals

$$2\sum_{k=1}^{n-1} k(n-k) = 2(\sum_{k=1}^{n-1} nk - \sum_{k=1}^{n-1} k^2) = 2\left(\frac{n^2(n-1)}{2} - \frac{n(n-1)(2n-1)}{6}\right) = \frac{n^3-n}{3}.$$

1 point.

We'll call any pair (u, v) such that u is white, v is black and u and v are in the same row or column a pretty pair. Now we will prove that the beauty of every convex board is less than or equal to the beauty of the Unicorn. We will do this by performing an algorithm which turns an arbitrary board into the Unicorn in finitely many steps.

Consider an arbitrary convex board. Let a_i be the number of black coloured cells in the i-th row. We perform the following algorithm:

If the board is equal to the Unicorn, we are done. Otherwise, find the first row in which $a_i \neq n+1-i$. Then, we consider two cases:

1. $a_i < n+1-i$. We colour the a_i+1-th cell in the i-th row black.

   1 point.

   We claim that the beauty of the board didn't decrease.
   We now count the number of black/white cells which are in the same row or column as the cell we colored and which are distinct from it.
   The number of black cells in the same row is equal to a_i, and the number of black cells in the same column is $i-1$. On the other hand, the number of white cells in the same row is $n-1-a_i$ and the number of white cells in the same column is $n-i$.

   1 point.

   Therefore, the difference of beauties of the board before and after coloring the a_i+1-th cell of i-th row black is $a_i + (i-1) - (n-1-a_i) - (n-i) = 2(a_i + i - n) \leqslant 0$, which implies that the new board's beauty is not smaller.

   1 point.

2. $a_i > n+1-i$. Let $j \geqslant i$ be the biggest index such that $a_j = a_i$. We colour the a_i-th cell of the j-th column white.

   1 point.

   We claim that the beauty of the board didn't decrease.
   As in the first case, we count the number of black/white cells which are in the same row or column as the cell we colored and which are distinct from it.
   The number of white cells in the same row equals $n-a_j$, and the number of white cells in the same column equals $n-j$. On the other hand, the number of black cells in the same row equals a_j-1, and the number of black cells in the same column equals $j-1$.

   1 point.

   Therefore, the difference of beauties of the board before and after coloring the a_j-th cell of j-th row white is $(n-a_j)+(n-j)-(a_j-1)-(j-1) = 2(n+1-j-a_j) \leqslant 2(n+1-i-a_i) < 0$, which implies that the new board's beauty is bigger.

   1 point.

The algorithm terminates because after each step, the number of positions where the board differs from the Unicorn decreases by 1. Therefore, the maximum beauty is achieved for the Unicorn.

1+1 points.

Second Solution. Consider an arbitrary convex board. Let a_i denote the number of black cells in the i-th row. Furthermore, we define $a_0 = n$ and $a_{n+1} = 0$. Then the number of pretty pairs (u, v) such that u and v are in the same row equals

$$\sum_{i=1}^{n} a_i(n - a_i).$$

1 point.

The number of columns with at least i black cells equals a_i.

1 point.

This implies that the number of columns with exactly i black cells equals the difference between the number of columns with at least i black cells and the number of columns with at least $i + 1$ black cells. Therefore, the number of columns with exactly i black cells equals $a_i - a_{i+1}$.

2 points.

This implies that the number of pretty pairs (u, v) such that u and v are in the same column equals

$$\sum_{i=1}^{n} (a_i - a_{i+1})i(n - i).$$

1 point.

Therefore, the beauty of the board equals

$$\sum_{i=1}^{n} a_i(n - a_i) + (a_i - a_{i+1})i(n - i) = \sum_{i=1}^{n} a_i(n - a_i + i(n - i) - (i - 1)(n + 1 - i)) = \sum_{i=1}^{n} a_i(2n + 1 - 2i - a_i).$$

1 point.

For a fixed $i \in \{1, \ldots, n\}$, $a_i(2n + 1 - 2i - a_i)$ is a quadratic function of a_i, which is increasing for $a_i \in [0, n - i + \frac{1}{2}]$ and decreasing for $a_i \in [n - i + \frac{1}{2}, n]$, and the maximum among all integer a_i is then achieved if $a_i \in \{n - i, n - i + 1\}$.

1 point.

Therefore, the whole sum is maximised if $a_i \in \{n - i, n - i + 1\}$ for all $i \in \{1, \ldots, n\}$.

1 point.

Any board with $a_i \in \{n - i, n - i + 1\}$ is convex since then $a_i \geqslant a_{i+1}$ for any of the possible choices.

1 point.

In this case, the sum equals

$$\sum_{i=1}^{n} (n - i)(n - i + 1) = \sum_{i=1}^{n} i(i - 1) = \sum_{i=1}^{n} i^2 - i = \frac{n(n + 1)(2n + 1) - 3n(n + 1)}{6} = \frac{n^3 - n}{6}.$$

1 point.

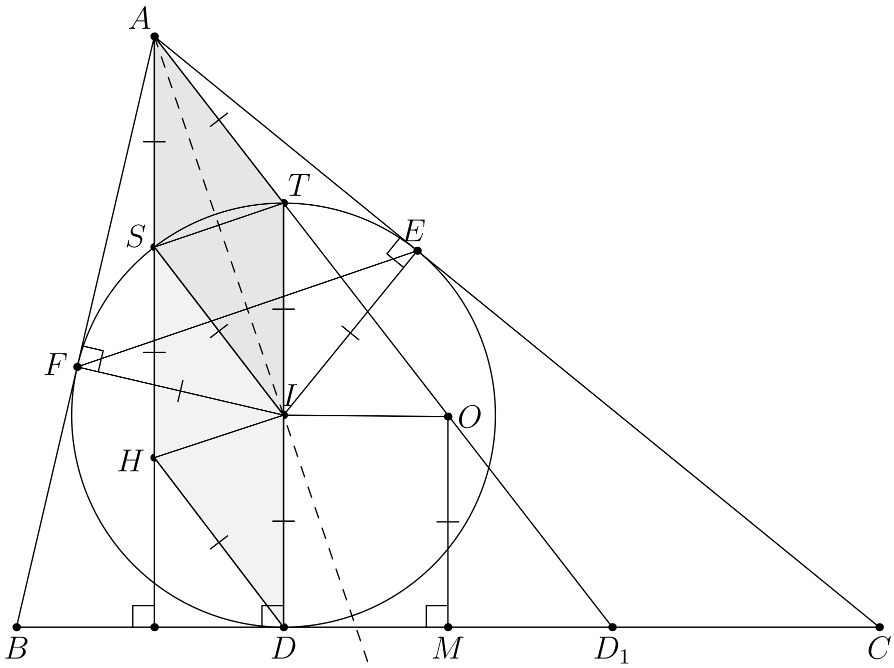
Notes on marking:

- A student is awarded the maximum of the two scores he gets by following either of the two marking schemes. Points from different solutions are not additive.
- If the student produces an optimal board, writes down its beauty, but does not simplify the expression to a closed form, then:
  a) if the student does not prove the optimality of the board (a "0+" solution), he is awarded **0 points** for this part;
  b) if the student proves the optimality of the board (a "10-" solution), he is awarded **1 point** for this part.
- In the **First Solution**, the last **2 points** are only awarded if he gives a correct algorithm.
- In the **First Solution**, if the student has a correct algorithm, but fails to prove that it terminates, he should be deducted **1 point**.
- In the **Second Solution**, the "other direction" is implicit in the last part of the solution. This is because the Unicorn configuration is covered by the given equality cases. If the student gives an optimal board as in the other solutions, and then shows that his optimal board is contained in the equality case, his solution is complete.
  However, if the student does not in any way show that his lower and upper bounds match, he should be deducted **1 point**.

Problem 3. In an acute triangle ABC with $|AB| \neq |AC|$, let I be the incenter and O the circumcenter. The incircle is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ in D, E and F respectively. Prove that if the line parallel to EF passing through I, the line parallel to AO passing through D and the altitude from A are concurrent, then the point of concurrence is the orthocenter of the triangle ABC.

(Petar Nizić-Nikolac)

Sketch.



Solution. Let H be that concurrence point. We shall prove that H is the orthocenter of the triangle ABC. Firstly we observe that $AFIE$ is a deltoid (because $|AE| = |AF|$ and $|IE| = |IF|$), so $AI \perp EF \parallel HI$.

1 point.

Using the fact that AI is the bisector of $\angle OAH$ and $AH \parallel ID$ we conclude that

$$\angle DIH = \angle AID - 90^\circ = 180^\circ - \angle IAH - 90^\circ = 90^\circ - \frac{\angle OAH}{2} = 90^\circ - \frac{\angle HDI}{2}$$

so triangle IHD is an isoscales one.

1 point.

Denote by T the second intersection of the line DI and the incircle and S as the point such that $SHDI$ is a rhombus. It follows that S lies on AH, but also that triangle ISH is an isoscales one, so

$$\angle SIA = 90^\circ - \angle SIH = 90^\circ - \angle SHI = 90^\circ - \angle HID = \angle AIT = \angle IAS.$$

Hence $|AS| = |SI| = |ID| = |IT|$ (we used that I is the midpoint of $\overline{TD}$), so $ASIT$ is a rhombus.

3 points.

Lemma. A, T, O and D_1 are collinear, where D_1 is the point where A-excircle is tangent to BC.
Proof. Firstly, A, T and O are collinear as $AT \parallel SI \parallel HD \parallel AO$.

1 point.

Secondly, A, T and D_1 are collinear as there is homothety from A sending incircle to A-excircle, so the "highest" points (w.r.t. $\overline{BC}$) of these circles (T and D_1) and the center of homothety (A) are collinear. Therefore, A, T, O and D_1 are collinear. □

1 point.

Denote by M the midpoint of $\overline{BC}$. We know that $|BD| = \frac{|AB|+|BC|-|AC|}{2} = |CD_1|$, so M is the midpoint of $\overline{DD_1}$.

1 point.

As TDD_1 is a right triangle and $\angle OMD_1 = 90^\circ$ we conclude that $\overline{OM}$ is a D_1-midline in the triangle TDD_1, hence

$$2|OM| = |TD| = |TI| + |ID| = |AS| + |SH| = |AH|.$$

1 point.

Now we can conclude in various ways (for example, using the Euler line argument) that H is the orthocenter of the triangle ABC.

1 point.

Notes on marking:

- Essentially, **5 points** are awarded for proving that $AHDT$ is a parallelogram with longer side being twice the size of the shorter side, next **4 points** are awarded for proving that $2|OM| = |AH|$ is true, and **1 point** is awarded for deduction that H is indeed an orthocenter.
- If a student states that A, T, D_1 are collinear in a general triangle without using it to prove the problem (for example, by introducing the point O and stating that it should be on the line), it should be awarded **0 points**. On the other hand, if a student uses this fact to prove the problem, it does not have to prove this fact and it is enough to state it. In that case it is awarded **1 point**.
- If a student states that $2|OM| = |AH|$ in a general triangle without using it to prove the problem (for example, by noting that $|OM| = |ID|$), it should be awarded **0 points**. On the other hand, if a student uses this fact to prove the problem, it does not have to prove this fact and it is enough to state it. In that case it is awarded **1 point**.
- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

for all $x, y \in \mathbb{R}$.

(Adrian Beker)

Solution. It is easily checked that $f(x) = x+1$ is a valid solution. We will prove that it is the only solution. Let $P(x, y)$ denote the assertion

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

and let $a = f(0)$. We will first prove the following claim:

Claim. f is injective

Proof: Suppose that $f(x) = f(y) = t$ for some $x, y \in \mathbb{R}$. We have:

$$P(x, x) \implies t + f(xt + t) = f(x + 2t) + x^2$$

$$P(x, y) \implies t + f(yt + t) = f(x + 2t) + xy$$

Subtracting the last two equations yields $f(xt+t)-f(yt+t) = x(x-y)$. Similarily, we have $f(yt+t)-f(xt+t) = y(y-x)$ which implies $(x - y)^2 = 0 \implies x = y$, hence f is injective. □

4 points.

We have:

$$P(x, 0) \implies f(x) + f(a) = f(x + 2a) \tag{1}$$

Setting $x = -a$ yields $f(-a) = 0$. Now we have:

$$P(x, -a) \implies f(-af(x)) = -ax \tag{2}$$

Again, setting $x = -a$ yields $a = a^2$, hence $a \in \{0, 1\}$.

1 point.

Case 1. $a = 0$

$$P(0, y) \implies f(f(y)) = f(2f(y))$$

Since f is injective, we have $f(y) = 2f(y) \implies f(y) = 0$ for all $y \in \mathbb{R}$, which is clearly impossible.

1 point.

Case 2. $a = 1$

Now (2) implies $f(-f(x)) = -x$ for all $x \in \mathbb{R}$ This means that f is bijective. On the other hand, (1) implies that $f(x) + f(1) = f(x + 2)$ for all $x \in \mathbb{R}$.

$$P(x + 2, y) \implies f(x + 2) + f(yf(x + 2) + f(y)) = f(x + 2 + 2f(y)) + (x + 2)y$$

$$f(x) + f(y) + f(yf(x) + f(y) + yf(1)) = f(x + 2f(y)) + f(1) + xy + 2y$$

1 point.

By subtracting the initial equation from this one, we obtain:

$$f(yf(x) + f(y) + yf(1)) = f(yf(x) + f(y)) + 2y$$

If $y \neq 0$, we can choose $x \in \mathbb{R}$ such that $f(x) = -\frac{f(y)}{y}$ because f is surjective, hence the last equation yields:

$$f(yf(1)) = 2y + 1$$

2 points.

for all $y \neq 0$, but it is also true for $y = 0$. In particular, setting $y = -\frac{1}{2}$ yields $f(-\frac{f(1)}{2}) = 0$. Since f is injective and $f(-1)$ =0, it follows that $f(1) = 2 \implies f(2y) = 2y + 1$ for all $y \in \mathbb{R}$. Finally, we deduce that $f(x) = x + 1$ for all $x \in \mathbb{R}$, as desired. □

1 point.

Notes on marking:

- The case $a = 1$ can be finished without injectivity. If a student deduces that f is linear and checks that the only option for f is $f(x) = x + 1$, he should get **1 point**.
- If a student manages to prove that f is injective in the case $a = 0$, he should get **4 points** from the first part of the solution since in the case $a = 1$ the proof can be finished without injectivity.
- If a student doesn't check that $f(x) = x + 1$ is indeed a solution or at least mention that it can be easily checked, he should lose **1 point**.