8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Junior Category

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Problems and Solutions

Problem 1. Every positive integer is marked with a number from the set $\{0, 1, 2\}$, according to the following rule:

if a positive integer k is marked with j, then the integer $k + j$ is marked with 0.

Let S denote the sum of marks of the first 2019 positive integers. Determine the maximum possible value of S.

(Ivan Novak)

First Solution. Consider an arbitrary marking scheme which follows the given rule.

Let a denote the number of positive integers from the set $\{1, \ldots, 2019\}$ which are marked with a 2, b the number of those marked with a 1, and c the number of those marked with a 0.

1 point.

We have $S = 2a + b$.

1 point.

For every positive integer $j \in \{1, \ldots, 2017\}$ which is marked with a 2, the number $j + 2$ is marked with a 0. This implies that the number of positive integers less than 2017 marked with 2 is less than or equal to c.

1 point.

Hence, this implies $a \leqslant c + 2$. We then have

$$S = 2a + b \leqslant a + b + c + 2 = 2019 + 2 = 2021.$$

3 points.

Consider the following marking scheme:

$$210|210|210|\underbrace{2200|2200|2200\ldots2200}_{\text{502 blocks of 2200}}|22|0000\ldots.$$

Here the i-th digit in the sequence denotes the mark of positive integer i. For this marking, $S = 2021$, and therefore 2021 is the maximum possible value of S.

4 points.

Second Solution. The marking scheme for which $S = 2021$ is the same as in the first solution.

4 points.

Let S_n denote the sum of marks of first n positive integers, and let a_k denote the mark of k. Without loss of generality we may assume $a_j = 0$ for all integers $j \leqslant 0$. We'll prove the following claim by strong mathematical induction:

for every positive integer n, $S_n \leqslant n + 2$ and if equality holds, then $a_n = 2$.

1 point.

The base cases for $n \in \{1, 2\}$ trivially hold. Suppose the claim is true for all $k \leqslant n$ for some $n \geqslant 2$.
Suppose there exists a marking scheme for which $S_{n+1} \geqslant n + 4$. Then if $a_{n+1} < 2$, we have $S_n \geqslant n + 3$, which is a contradiction. Hence, $a_{n+1} = 2$.

1 point.

This implies that $a_n \in \{0, 2\}$. If $a_n = 0$, then $S_{n-1} \geqslant n + 2$, which is a contradiction. So, $a_n = 2$.

1 point.

Now $a_{n-1} = 0$ because both a_n and a_{n+1} are nonzero. We now have $S_{n-2} \geqslant n$, and by the induction hypothesis, it must hold that $S_{n-2} = n$ and $a_{n-2} = 2$. However, this is in contradiction with a_n being nonzero. Hence, $S_{n+1} \leqslant n + 3$.

1 point.

Suppose $S_{n+1} = n + 3$ and $a_{n+1} \neq 2$. If $a_{n+1} = 0$, then $S_n \geqslant n + 3$, which is a contradiction. Thus, $a_{n+1} = 1$.

1 point.

Then $S_n = n + 2$, which implies $a_n = 2$. Then we must have $a_{n-1} = 0$, and then $S_{n-2} = n$, which implies $a_{n-2} = 2$, but a_n is nonzero, which is a contradiction. Therefore, the claim is true for $n + 1$, which implies it is true for all positive integers. In particular, $S_{2019} \leqslant 2021$, which combined with the construction implies that the maximum value of S is 2021.

1 point.

Notes on marking:

- If a student forgets to write additional zeros beyond the first 2019 digits in his construction, but the construction is otherwise valid, he should be awarded all **4 points** for this part.
- There are many different optimal marking schemes. For example, $2200|210|210|\ldots|210|22|000\ldots$, where the block $|210|$ repeats 671 times.
- In the **Second Solution**, if the student writes only the first part of the induction hypothesis without the assumption that $a_n = 2$ in the case of equality: he should be awarded **0 points**, unless he reaches additional conclusions which lead to the solution.
- In the **Second Solution**, if the student doesn't comment on the base case/cases at all, he should be deducted **1 point**.
- If the student proves any nontrivial lemma useful for any of the solutions, but the lemma itself isn't worth any points and the student wouldn't otherwise get any of the **6 points** given for proving the bound, he should get **1 point** for this part.

Problem 2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined recursively such that $x_1 = \sqrt{2}$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \text{ for } n \in \mathbb{N}.$$

Prove that the following inequality holds:

$$\frac{x_1^2}{2x_1x_2 - 1} + \frac{x_2^2}{2x_2x_3 - 1} + \ldots + \frac{x_{2018}^2}{2x_{2018}x_{2019} - 1} + \frac{x_{2019}^2}{2x_{2019}x_{2020} - 1} > \frac{2019^2}{x_{2019}^2 + \frac{1}{x_{2019}^2}}.$$

(Ivan Novak)

First Solution. Notice that by squaring the assertion $x_{n+1} = x_n + \frac{1}{x_n}$ we obtain the equality $x_{n+1}^2 = x_n^2 + \frac{1}{x_n^2} + 2 \implies x_n^2 + \frac{1}{x_n^2} = x_{n+1}^2 - 2$, which implies that the right hand side equals $\dfrac{2019^2}{x_{2020}^2 - 2}$.

1 point.

On the other hand, we have

$$2x_nx_{n+1} - 1 = 2x_n(x_n + \frac{1}{x_n}) - 1 = 2x_n^2 + 1.$$

1 point.

This implies that the sum on the left hand side can be written as

$$\frac{1}{2 + \frac{1}{x_1^2}} + \frac{1}{2 + \frac{1}{x_2^2}} + \ldots + \frac{1}{2 + \frac{1}{x_{2019}^2}}$$

1 point.

By squaring the given assertion, we get the equality $2 + \frac{1}{x_n^2} = x_{n+1}^2 - x_n^2$. This implies that the left hand side equals

$$\frac{1}{x_2^2 - x_1^2} + \frac{1}{x_3^2 - x_2^2} + \ldots + \frac{1}{x_{2019}^2 - x_{2018}^2} + \frac{1}{x_{2020}^2 - x_{2019}^2}.$$

1 point.

Using the inequality between arithmetic and harmonic mean, we find that the left hand side is greater than or equal to

$$\frac{2019^2}{(x_2^2 - x_1^2) + (x_3^2 - x_2^2) + \ldots + (x_{2020}^2 - x_{2019}^2)}.$$

4 points.

We now notice that the denominator is a telescoping sum and it equals $x_{2020}^2 - x_1^2$, which implies the right hand side equals

$$\frac{2019^2}{x_{2020}^2 - x_1^2} = \frac{2019^2}{x_{2020}^2 - 2},$$

which is exactly equal to the right hand side.

1 point.

The equality cannot hold because $x_2^2 - x_1^2 \neq x_3^2 - x_2^2$.

1 point.

Second Solution. As in the first solution, we obtain that the left hand side equals

$$\frac{1}{2+\frac{1}{x_1^2}}+\frac{1}{2+\frac{1}{x_2^2}}+\ldots+\frac{1}{2+\frac{1}{x_{2018}^2}}+\frac{1}{2+\frac{1}{x_{2019}^2}}.$$

2 points.

Using the inequality between arithmetic and harmonic mean, we get that the left hand side is greater than or equal to

$$\frac{2019^2}{2\cdot 2019+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{2019}^2}}.$$

4 points.

We now prove by mathematical induction that

$$2\cdot n+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{n-1}^2}=x_n^2$$

holds for every $n\in\mathbb{N}$.

1 point.

For $n=1$, we have $2\cdot 1=\sqrt{2}^2$. Suppose the claim is true for some $n\in\mathbb{N}$. Then

$$x_{n+1}^2=2+x_n^2+\frac{1}{x_n^2}=2+2n+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{n-1}^2}+\frac{1}{x_n^2},$$

where we used the induction hypothesis for the last equality. This proves the claim.

2 points.

In particular, for $n=2019$, we have that

$$\frac{2019^2}{2\cdot 2019+\frac{1}{x_1^2}+\frac{1}{x_2^2}+\ldots+\frac{1}{x_{2019}^2}}=\frac{2019^2}{x_{2019}^2+\frac{1}{x_{2019}^2}},$$

which proves the inequality.

The equality cannot hold because $\frac{1}{x_1^2}+2\neq\frac{1}{x_2^2}+2$.

1 point.

Third Solution. We prove by mathematical induction that for every $n \geqslant 2$ the following inequality holds:

$$\frac{x_1^2}{2x_1x_2-1}+\frac{x_2^2}{2x_2x_3-1}+\ldots+\frac{x_n^2}{2x_nx_{n+1}-1}>\frac{n^2}{x_n^2+\frac{1}{x_n^2}}.$$

For $n=2$, the left hand side equals $\frac{2}{5}+\frac{4.5}{10}=\frac{17}{20}$, and the right hand side equals $\frac{4}{\frac{9}{2}+\frac{2}{9}}=\frac{72}{85}<\frac{17}{20}$, which proves the base case.

Suppose the claim was true for some $n \in \mathbb{N}$. Then by the induction hypothesis, we know that

$$\frac{x_1^2}{2x_1x_2-1}+\frac{x_2^2}{2x_2x_3-1}+\ldots+\frac{x_n^2}{2x_nx_{n+1}-1}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}>\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}.$$

It suffices to prove that

$$\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}\geqslant\frac{(n+1)^2}{x_{n+1}^2+\frac{1}{x_{n+1}^2}}.$$

1 point.

We now prove that $2x_{n+1}x_{n+2}-1=2x_{n+1}^2+1$ as in the first solution.

1 point.

We then have

$$\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}x_{n+2}-1}=\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{x_{n+1}^2}{2x_{n+1}^2+1}=\frac{n^2}{x_n^2+\frac{1}{x_n^2}}+\frac{1}{2+\cfrac{1}{x_{n+1}^2}}.$$

1 point.

By the inequality of arithmetic and harmonic mean, this is greater than or equal to

$$\frac{(n+1)^2}{x_n^2+\frac{1}{x_n^2}+2+\frac{1}{x_{n+1}^2}}.$$

5 points.

Notice that squaring the assertion $x_{n+1}=x_n+\frac{1}{x_n}$, we obtain

$$x_n^2+\frac{1}{x_n^2}+2=x_{n+1}^2.$$

1 point.

This implies that

$$\frac{(n+1)^2}{x_n^2+\frac{1}{x_n^2}+2+\frac{1}{x_{n+1}^2}}=\frac{(n+1)^2}{x_{n+1}^2+\frac{1}{x_{n+1}^2}},$$

which is exactly equal to the right hand side. Therefore, the claim is proven by the principle of mathematical induction. In particular, the claim is true for $n=2019$, which proves the inequality.

1 point.

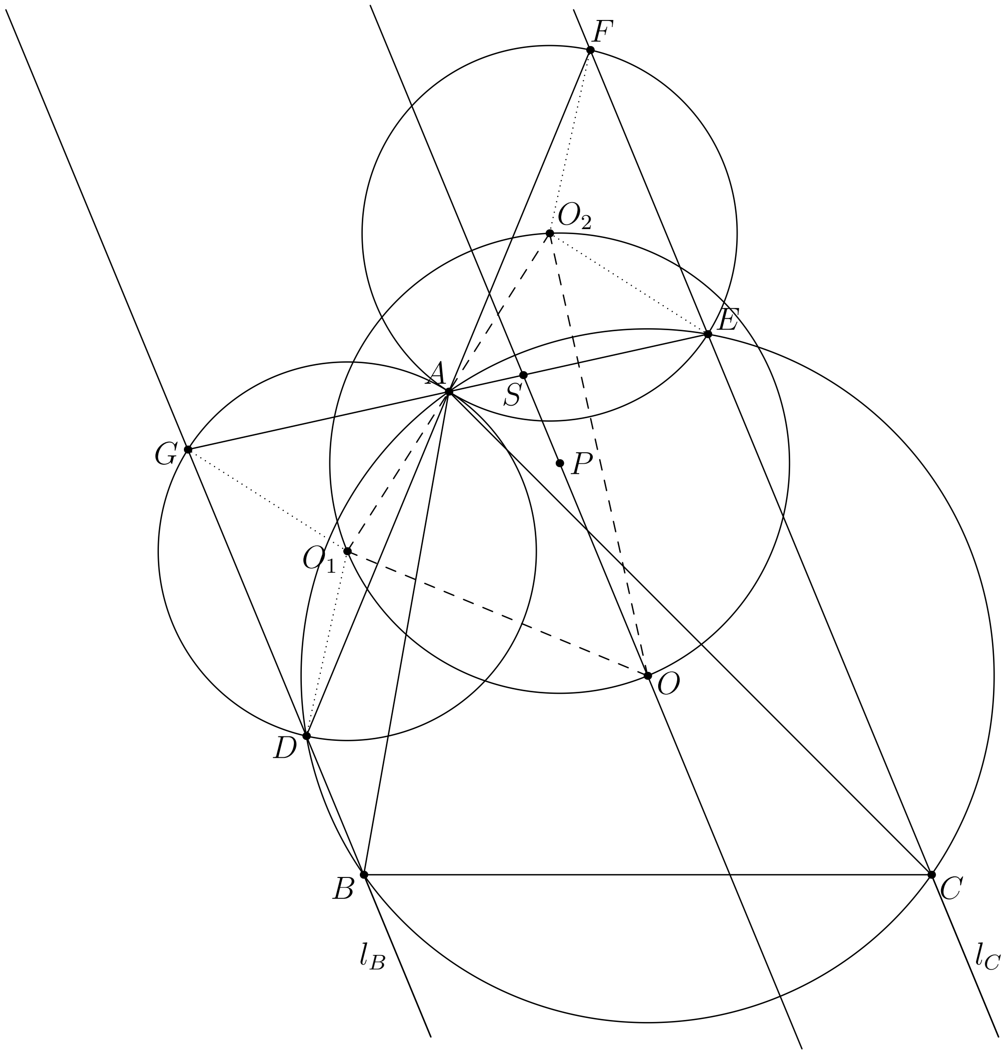
Notes on marking:

- Points from separate solutions can not be added. The student should be awarded the maximum of the points scored in the 3 presented solutions, or an appropriate number of points on an alternative solution.
- The third solution gives **5 points** for the use of AM-HM inequality as opposed to **4 points** in the first solution because in the third solution it is not necessary to comment the equality case. However, if a student has $n=1$ as a basis of induction and doesn't comment the equality case, he should be deducted **1 point** out of possible **5**.
- The point for proving that the equality cannot be achieved is only awarded if the student has proved the non-strict version of inequality.

Problem 3. Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$.

(Stefan Lozanovski)

Sketch.



Solution. Let us write $\angle BAC = \alpha, \angle ABC = \beta, \angle ACB = \gamma$.

Lemma. Triangles AGD and AEF are similar to the triangle ABC.
Proof. As $DBCAE$ is a cylic pentagon we have

$$\angle GDA = \angle BCA = \gamma.$$

1 point.

Now from $l_B \parallel l_C$ we get that

$$\angle DBA = \angle DBC - \beta = 180^\circ - \angle BCE - \beta = \alpha + \gamma - \angle BCE = \alpha - \angle ACE$$

1 point.

so from the cyclicity

$$\angle BCD = \angle BAD = 180^\circ - \angle DBA - \angle ADB = 180^\circ - (\alpha - \angle ACE) - (180^\circ - \gamma) = \gamma - \alpha + \angle ACE$$

1 point.

Hence

$$\angle DAG = \angle DCE = \angle BCA - \angle BCD + \angle ACE = \alpha$$

1 point.

Therefore AGD is similar to the triangle ABC, and similarly for AEF. □

Now as G, A and E are collinear and F, A and D are collinear, using *Lemma* we get that O, O_1 and O_2 are collinear.

1 point.

As O_1 is the circumcenter of the triangle ADG and O_1D is the bisector of the chord $\overline{AD}$ we get that

$$\angle AO_1O = \frac{1}{2}\angle AO_1D = \angle AGD = \beta$$

and similarly $\angle AO_1O = \gamma$, so the triangle OO_1O_2 is similar to the triangle ABC.

2 points.

Now as P is the circumcenter of the triangle OO_1O_2 from the previous similarity we get that

$$\angle O_1OP = \angle BAO$$

1 point.

Hence

$$\angle DOP = \angle DOO_1 + \angle O_1OP = \angle DBA + \angle BAO = \angle DBA + \angle ABO = \angle DBO = \angle ODB$$

so $l_B \parallel OP \parallel l_C$.

2 points.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Let u be a positive rational number and m be a positive integer. Define a sequence $q_1, q_2, q_3, \ldots$ such that $q_1 = u$ and for $n \geqslant 2$:

$$\text{if } q_{n-1} = \frac{a}{b} \text{ for some relatively prime positive integers } a \text{ and } b, \text{ then } q_n = \frac{a+mb}{b+1}.$$

Determine all positive integers m such that the sequence $q_1, q_2, q_3, \ldots$ is eventually periodic for any positive rational number u.

Remark: A sequence $x_1, x_2, x_3, \ldots$ is *eventually periodic* if there are positive integers c and t such that $x_n = x_{n+t}$ for all $n \geqslant c$.

(Petar Nizić-Nikolac)

Solution. We will prove that the sequence is eventually periodic if and only if m is odd.
Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be sequences of numerators and denumerators of $q_1, q_2, q_3, \ldots$ respectively when written in the irreducible form, i.e. for $n \in \mathbb{N}$:

$$q_n = \frac{a_n}{b_n} \qquad gcd(a_n, b_n) = 1$$

Say that there was *reduction in the n^{th} step* if $\gcd(a_n + mb_n, b_n + 1) > 1$.

Case 1. m is even
Set $u = \frac{1}{1}$. Assume for the sake of contradiction that $q_1, q_2, q_3, \ldots$ is eventually periodic. Then $(b_n)_{n\in\mathbb{N}}$ is bounded so there is $r > 1$ (pick the smallest one) such that there was reduction in the r^{th} step. Easy to see that

$$q_1 = \frac{1}{1}, q_2 = \frac{m+1}{2}, q_3 = \frac{3m+1}{3}, q_4 = \frac{6m+1}{4}, q_5 = \frac{10m+1}{5}, \ldots, q_r = \frac{\binom{r}{2}m+1}{r}$$

2 points.

Now as m is even we have

$$\gcd\left(a_r + mb_r, b_r + 1\right) = \gcd\left(\binom{r}{2}m + 1 + mr, r+1\right) = \gcd\left(\binom{r+1}{2}m + 1, r+1\right) = \gcd\left((r+1)r\frac{m}{2} + 1, r+1\right) = 1$$

so this is a contradiction, and hence it is not eventually periodic for any positive rational number u.

1 point.

Case 2. m is odd
Assume that there is $r \in \mathbb{N}$ such that there was no reduction in the steps r, $r+1$, $r+2$ and $r+3$. Then for $i \in \{1, 2\}$:

$$(a_{r+i+2}, b_{r+i+2}) \equiv (a_{r+i} + mb_{r+i} + mb_{r+i+1}, b_{r+i} + 1 + 1) \equiv (a_{r+i} + 2mb_{r+i} + m, b_{r+i} + 2) \equiv (a_{r+i} + 1, b_{r+i}) \pmod 2$$

so at least one of the following pairs: $(a_{r+1}, b_{r+1}), (a_{r+2}, b_{r+2}), (a_{r+3}, b_{r+3}), (a_{r+4}, b_{r+4})$ has both even entries which is impossible (as they are coprime). Hence there was at least one reduction in the steps r, $r+1$, $r+2$ and $r+3$.

2 points.

Therefore for all $n \geqslant 1$:

$$\max\{b_{n+1}, b_{n+2}, b_{n+3}, b_{n+4}\} \leqslant \min\{b_{n+1}, b_{n+2}, b_{n+3}, b_{n+4}\} + 3 \leqslant \frac{1}{2}\max\{b_n, b_{n+1}, b_{n+2}, b_{n+3}\} + 3$$

so there exists $C \geqslant 1$ such that $b_n \leqslant 6$ for all $n \geqslant C$.

2 points.

Similarly for all $n \geqslant C$:

$$\max\{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} \leqslant \min\{a_{n+1}, a_{n+2}, a_{n+3}, a_{n+4}\} + 3 \cdot 6m \leqslant \frac{1}{2}\max\{a_n, a_{n+1}, a_{n+2}, a_{n+3}\} + 18m$$

so there exists $D \geqslant C$ such that $a_n \leqslant 36m$ for all $n \geqslant D$.

2 points.

We conclude that for all $n \geqslant D$ there are finitely many pairs $(6 \cdot 36m = 216m)$ that (a_n, b_n) attains so it becomes eventually periodic for any positive rational number u.

1 point.

Notes on marking:

- **Case 1** and **Case 2** are always worth **3 points** and **7 points** respectively.