7$^{\text{TH}}$ EUROPEAN MATHEMATICAL CUP

8th December 2018 - 16th December 2018

Junior Category

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Problems and Solutions

Problem 1. Let a, b, c be non-zero real numbers such that

$$a^2 + b + c = \frac{1}{a},$$
$$b^2 + c + a = \frac{1}{b},$$
$$c^2 + a + b = \frac{1}{c}.$$

Prove that at least two of a, b, c are equal.

(Daniel Paleka)

First Solution. Let's assume the opposite, i.e. a, b and c are pairwise non-equal. By subtracting the second equality from the first one, we obtain

$$\begin{aligned}
(a^2 + b + c) - (b^2 + c + a) &= \frac{1}{a} - \frac{1}{b} \\
(a^2 - b^2) + (b - a) &= \frac{b - a}{ab} \\
(a - b)(a + b) - (a - b) + \frac{a - b}{ab} &= 0 \\
(a - b)\left(a + b - 1 + \frac{1}{ab}\right) &= 0
\end{aligned}$$

1 point.

Since $a \neq b$ we may conclude

$$a + b - 1 + \frac{1}{ab} = 0 \tag{1}$$

1 point.

Similarly, subtracting the third equality from the second one, combined with $b \neq c$, gives us

$$b + c - 1 + \frac{1}{bc} = 0 \tag{2}$$

1 point.

Expressions on the left side in (1) and (2) are both equal to 0 which specifically implies

$$\begin{aligned}
a + b - 1 + \frac{1}{ab} &= b + c - 1 + \frac{1}{bc} \\
(a - c) + \frac{1}{b} \cdot \left(\frac{1}{a} - \frac{1}{c}\right) &= 0 \\
(a - c) + \frac{1}{b} \cdot \frac{c - a}{ac} &= 0 \\
(a - c)\left(1 - \frac{1}{b} \cdot \frac{1}{ac}\right) &= 0 \\
1 - \frac{1}{abc} &= 0 \\
abc &= 1
\end{aligned}$$

2 points.

Inserting that back into (1) results with

$$0 = a + b - 1 + \frac{1}{ab} = a + b - 1 + \frac{abc}{ab} = a + b - 1 + c$$

$$\Rightarrow\ b + c = 1 - a \tag{3}$$

2 points.

Finally, combining (3) with the first of the 3 given equations results with

$$\begin{aligned} a^2 + b + c &= \frac{1}{a} \\ a^2 + 1 - a &= \frac{1}{a} \\ (a^2 - a) + \left(1 - \frac{1}{a}\right) &= 0 \\ a(a-1) + \frac{a-1}{a} &= 0 \\ (a-1)\left(a + \frac{1}{a}\right) &= 0 \\ (a-1) \cdot \frac{a^2+1}{a} &= 0 \end{aligned}$$

Because of $a^2 + 1 > 0$ we obtain $a - 1 = 0$, i.e. $a = 1$.

2 points.

Analogously we also find $b = c = 1$ which is a contradiction with the assumption so we conclude that at least two of a, b, c are equal.

1 point.

Second Solution. Let's assume the contrary, i.e. a, b and c are pairwise different. After multiplying the first equation with a, the second with b, and the third with c, we get:

$$a^3 + ab + ac = b^3 + bc + ba = c^3 + ca + cb = 1.$$

0 points.

In particular, the first two expressions are equal. Subtracting them and factorizing leads to:

$$a^3 - b^3 + ac - bc = 0$$
$$(a - b)(a^2 + ab + b^2 + c) = 0$$

1 point.

Since $a \neq b$ we may conclude:

$$a^2 + ab + b^2 + c = 0$$

1 point.

Similarly, we can get the same thing for b and c:

$$b^2 + bc + c^2 + a = 0$$

1 point.

Subtracting these two equations yields:

$$a^2 - c^2 + ab - bc + c - a = 0$$
$$(a - c)(a + c + b - 1) = 0$$

2 points.

Because $a \neq c$, we obtain:

$$a + b + c = 1$$

2 points.

Now we proceed to arrive to a contradiction in the same way as in the previous solution.

3 points.

Notes on marking:

- After obtaining $a = 1$, we may use that fact in (3) to conclude $b + c = 0$, i.e. $c = -b$. That gives us

  $$1 = abc = 1 \cdot b \cdot (-b) = -b^2$$

  which isn't satisfied for any $b \in \mathbb{R}$. Again we reach contradiction with the assumption and conclude that at least two of a, b, c are equal. This part of the solution should be awarded with **1 point.**

Problem 2. Find all pairs (x, y) of positive integers such that

$$xy \mid x^2 + 2y - 1.$$

(Ivan Novak)

First Solution. Notice that the condition implies

$$x \mid 2y - 1.$$

1 point.

This implies that there exists a positive integer k such that $kx = 2y - 1$, so $y = \frac{kx+1}{2}$.

1 point.

Returning to the starting assertion, we get that

$$\frac{x(kx+1)}{2} \mid x^2 + kx \implies kx + 1 \mid 2(k + x).$$

2 points.

For all positive integers k, x, the following inequality is satisfied, with equality if and only if $k = 1$ or $x = 1$:

$$\frac{2(k+x)}{kx+1} \leqslant 2 \iff 2(k-1)(x-1) \geqslant 0.$$

2 points.

So as $\frac{2(k+x)}{kx+1} \in \mathbb{N}$, then we conclude that $\frac{2(k+x)}{kx+1} \in \{1, 2\}$.

1 point.

We now have two possible cases.

1. $k + x = kx + 1$. In this case, $k = 1$ or $x = 1$.
   (a) If $x = 1$, then y can be any positive integer.
   (b) If $k = 1$, then $x = 2y - 1$, where y is any positive integer.

1 point.

2. $2k + 2x = kx + 1$. Then $2x - 1 = k(x - 2) \implies x - 2 \mid 2x - 1 \implies x - 2 \mid 3$. This has only two solutions, both of which are true by an easy check: $x = 3, k = 5, y = 8$ or $x = 5, k = 3, y = 8$.

2 points.

Therefore, the set of solutions is

$$(x, y) \in \{(1, t), (2t - 1, t), (3, 8), (5, 8) \mid t \in \mathbb{N}.\}$$

Second Solution. Let (x, y) be a solution, and let $\frac{x^2+2y-1}{xy} = g$. This equation is equivalent with $x^2-(gy)x+2y-1 = 0$. We know x is one root of the equation. Let R be the other root. Using Vieta's formulas, we obtain the following system of equations:

$$x + R = gy$$
$$xR = 2y - 1.$$

3 points.

From the first equation we get that R is an integer, and from the second equation we get that it is a positive integer.

1 point.

Using the same inequality as in Solution 1, we get that $gy \leqslant 2y$, which implies $g = 1$ or $g = 2$.

3 points.

We now split into two cases:

1. If $g = 1$, then $x^2 + 2y - 1 = xy \implies x^2 - 1 = y(x-2) \implies x - 2 \mid x^2 - 1 \implies x - 2 \mid 3$. This has only two solutions, both of which are true by an easy check: $x = 5, y = 8$ or $x = 3, y = 8$.

2 points.

2. If $g = 2$, then $x^2 + 2y - 1 = 2xy \implies x^2 - 1 = 2y(x-1) \implies x = 1$ or $x = 2y - 1$, and y can be any positive integer.

1 point.

Therefore, the set of solutions is

$$(x, y) \in \{(1, t), (2t - 1, t), (3, 8), (5, 8) \mid t \in \mathbb{N}.\}$$

Third Solution. Let (x, y) be a solution, and let $\frac{x^2+2y-1}{xy} = g$. This equation is equivalent with $x^2-(gy)x+2y-1 = 0$.

1 point.

The condition of the problem is satisfied if and only if the discriminant is a square of a positive integer.

1 point.

Let the discriminant be equal to k^2. Then

$$k^2 = g^2y^2 - 8y + 4 \iff k^2 = (gy-2)^2 + 4gy - 8y \iff$$
$$(k - gy + 2)(k + gy - 2) = 4y(g - 2).$$

2 points.

We now split into 3 cases depending on the size of g.

1. If $g > 2$, then $4y(g-2) > 0$ and $k + gy - 2 > 0$, which implies

$$k - gy + 2 > 0.$$

1 point.

Adding $2gy - 4y = 2y(g-2)$ to the both sides, we obtain

$$k + gy - 2 \geqslant k + gy + 2 - 4y > 2y(g-2) = \frac{(k+gy-2)(k-gy+2)}{2} \implies$$
$$k - gy + 2 < 2,$$

so the only possibility is $k - gy + 2 = 1$, $k + gy - 2 = 4y(g-2)$.
However, summing up the two equalities yields $2k = 4y(g-2)+1$, which is a contradiction modulo 2. Therefore, there are no solutions in this case.

2 points.

2. If $g = 1$, then $x^2 + 2y - 1 = xy \implies x^2 - 1 = y(x-2) \implies x - 2 \mid x^2 - 1 \implies x - 2 \mid 3$. This has only two solutions, both of which are true by an easy check: $x = 5, y = 8$ or $x = 3, y = 8$.

2 points.

3. If $g = 2$, then $x^2 + 2y - 1 = 2xy \implies x^2 - 1 = 2y(x-1) \implies x = 1$ or $x = 2y - 1$, and y can be any positive integer.

1 point.

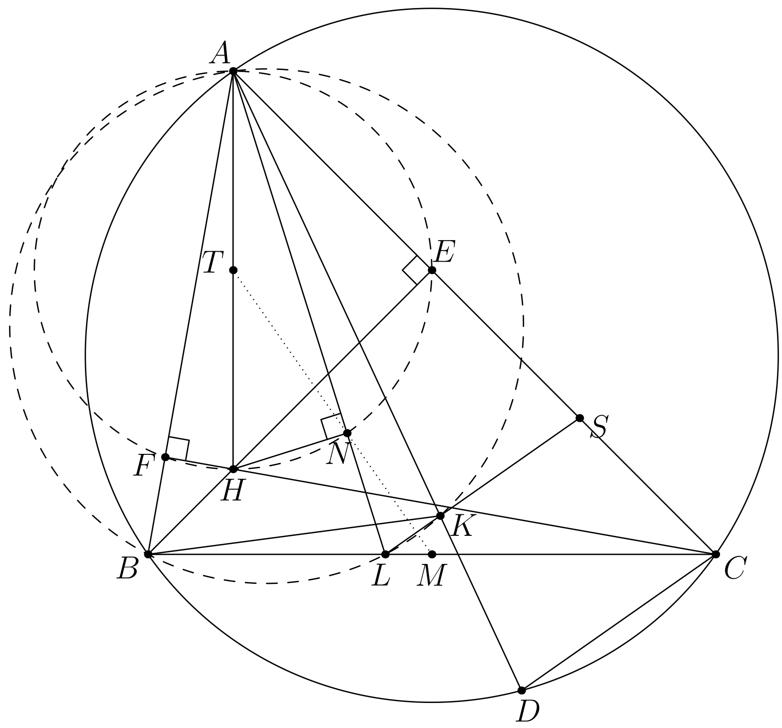
Notes on marking:

- Points from separate solutions can not be added. The competitor should be awarded the maximum of the points scored in the 2 presented solutions, or an appropriate number of points on an alternative solution.

Problem 3. Let ABC be an acute triangle with $|AB| < |AC|$ and orthocenter H. The circle with center A and radius $|AC|$ intersects the circumcircle of $\triangle ABC$ at point D and the circle with center A and radius $|AB|$ intersects the segment $\overline{AD}$ at point K. The line through K parallel to CD intersects BC at the point L. If M is the midpoint of $\overline{BC}$ and N is the foot of the perpendicular from H to AL, prove that the line MN bisects the segment $\overline{AH}$.

(Miroslav Marinov)

First Sketch.



First Solution. We start with the following:

Lemma 1. AL is the angle bisector of $\angle BAC$.

Proof: Since A, B, C and D lie on the same circle we obtain that $\angle ABC = \angle ADC = \angle ACD$.

1 point.

From that we get the following three equations:

$$\angle CAD = 180^\circ - 2\angle ADC = 180^\circ - 2\angle ABC$$

1 point.

$$\angle BCD = \angle BAD = \angle BAK = \angle BAC - (180^\circ - 2\angle ABC) = \angle ABC - \angle ACB$$

1 point.

$$\angle ABK = \angle AKB = 90^\circ - \frac{\angle BAK}{2} = 90^\circ - \frac{\angle ABC - \angle ACB}{2}$$

1 point.

Next from $LK \parallel CD$ it follows that $\angle CLK = \angle LCD = \angle BCD = \angle BAK$ so A, B, L and K are concyclic.

1 point.

Now we have

$$\angle BAL = \angle BAK - \angle LAK = \angle BAK - \angle LBK = (\angle ABC - \angle ACB) - (\angle ABC - \angle ABK) = \frac{\angle BAC}{2}$$

hence AL is the angle bisector of $\angle BAC$. □

1 point.

Let E and F be the feet of the altitudes from B and C in $\triangle ABC$. Observe that $\angle AEH = \angle AFH = \angle ANH = 90^\circ$ so A, E, H, N and F lie on the circle with diameter $\overline{AH}$.

1 point.

Since AL is the angle bisector of $\angle BAC$ its follows that $|NE| = |NF|$.

1 point.

Denote by T the midpoint of $\overline{AH}$. Since T is the circumcenter of $AEHNF$ we get $|TE| = |TF|$.

1 point.

Also since $\angle BEC = \angle CFB$, E and F lie on the circle with diameter $\overline{BC}$ from where we get $|ME| = |MF|$ so M, N and T lie on the bisector of $\overline{EF}$.

1 point.

Alternative proof of Lemma 1.

Similarly as in the first proof, we obtain that $\angle ABC = \angle ADC = \angle ACD$.

1 point.

Let S be the intersection of KL and AC. Since $LS \parallel DC$, we have $\angle ASL = \angle ACD = \angle ABC = \angle ABL$.

2 points.

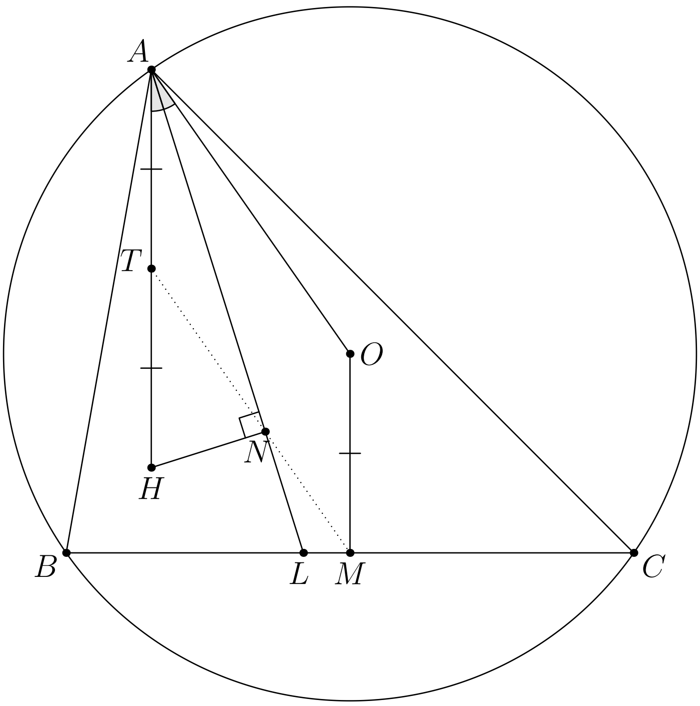
We also get $\angle AKS = \angle ADC = \angle ASK$, hence $|AS| = |AK| = |AB|$.

1 point.

Since B, L, S are not collinear (this is since SL is parallel to CD, which in turn isn't parallel to BC since $|AB| < |AC|$), we may conclude that $\triangle ABL$ and $\triangle ASL$ are congruent. The claim now follows. ☐

2 points.

Second Sketch.



Second Solution. We get similarly as in the **First Solution** that AL is the angle bisector of $\angle BAC$.

6 points.

Denote by T the midpoint of $\overline{AH}$. As $\triangle HNA$ is right-angled, we have that $\angle NTH = 2\angle NAH$.

1 point.

Denote by O the circumcenter of $\triangle ABC$. It is known that (as a consequence of existence of Euler line)

$$|AH| = 2|OM| \implies |AT| = \frac{|AH|}{2} = |OM|$$

and as AT and OM are both orthogonal to BC, they are parallel, so $ATMO$ is an parallelogram.

1 point.

Now as $\angle HAB = 90^\circ - \angle ABC = \angle OAC$, we know that AL is the angle bisector of OAH.

1 point.

Then we have that $\angle MTH = \angle OAH = 2\angle NAH = \angle NTH$ and we conclude that T, N and M are colinear.

1 point.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 4. Let n be a positive integer. Ana and Banana are playing the following game:

First, Ana arranges $2n$ cups in a row on a table, each facing upside-down. She then places a ball under a cup and makes a hole in the table under some other cup. Banana then gives a finite sequence of commands to Ana, where each command consists of swapping two adjacent cups in the row.

Her goal is to achieve that the ball has fallen into the hole during the game. Assuming Banana has no information about the position of the hole and the position of the ball at any point, what is the smallest number of commands she has to give in order to achieve her goal?

(Adrian Beker)

First Solution. We claim that the minimum number of commands is $n(3n-2)$.

Call a finite sequence of commands *valid* if it results in the ball falling into the hole while performing the commands, regardless of the inital position of the ball and the position of the hole. Also call a position an *endpoint* if it is either the first or the last position in the row.

Lemma 1. A sequence of commands is valid if and only if it results in each cup visiting both endpoints.

Proof. Suppose there exists a cup c that hasn't visited an endpoint p. Then the ball fails to fall into the hole in the case when it is under c and the hole is at p. Hence, the sequence is not valid. Conversely, if each cup has visited both endpoints, by discrete continuity it must have also visited all positions in between. Hence, the ball has certainly fallen into the hole, so the sequence is valid. □

2 points.

Now consider a valid sequence of commands. We will show that it has length at least $n(3n-2)$. Let C be the set of cups. For each $c \in C$, let k_c be the total number of commands involving c. Since each command involves two cups, the total number of commands is $\frac{1}{2}\sum_{c\in C} k_c$. So it suffices to show that $\sum_{c\in C} k_c \geqslant 2n(3n-2)$.

1 point.

For each $c \in C$, let x_c be the number of commands involving c before its first visit to an endpoint. Similarly, let y_c be the number of commands involving c after its last visit to an endpoint. Since c visited both endpoints, the number of commands between its first and last visit to an endpoint must be at least $2n-1$. Hence, $k_c \geqslant x_c + y_c + 2n - 1$.

2 points.

Now for each $1 \leqslant i \leqslant 2n$, let a_i be the cup at the i-th position from the left in the initial arrangement and similarly let b_i be the cup at the i-th position in the final arrangement. Then it follows that $x_{a_i}, y_{b_i} \geqslant \min(i-1, 2n-i)$ for all $1 \leqslant i \leqslant 2n$. Hence

$$\sum_{c\in C} x_c = \sum_{i=1}^{2n} x_{a_i} \geqslant \sum_{i=1}^{2n} \min(i-1, 2n-i) = n(n-1),$$

$$\sum_{c\in C} y_c = \sum_{i=1}^{2n} y_{b_i} \geqslant \sum_{i=1}^{2n} \min(i-1, 2n-i) = n(n-1),$$

$$\sum_{c\in C} k_c \geqslant \sum_{c\in C}(x_c + y_c + 2n - 1) \geqslant n(n-1) + n(n-1) + 2n(2n-1) = 2n(3n-2),$$

as desired. It remains to exhibit a valid sequence consisting of $n(3n-2)$ commands.

2 points.

Lemma 2. Consider n cups in a row. Then there is a sequence of $\frac{n(n-1)}{2}$ commands resulting in each cup visiting the first position (and similarly for the last position).

Proof. For each $1 \leqslant i \leqslant n$ in increasing order, for each $1 \leqslant j < i$ in decreasing order, swap the cups currently at the j-th and $(j+1)$-st positions. This clearly results in each cup visiting the first position and consist of $\sum_{i=1}^{n}(i-1) = \frac{n(n-1)}{2}$ commands, as desired (the case for the last position is analogous). □

Corollary. For $2n$ cups in a row, there is a sequence of $n(n-1)$ steps resulting in each of the first n cups visiting the first position and each of the last n cups visiting the last position.

2 points.

Now first apply the algorithm from the corollary. Then for each $1 \leqslant i \leqslant n$ in decreasing order, for each $0 \leqslant j < n$ in increasing order, swap the cups currently at the $(i+j)$-th and $(i+j+1)$-st positions. Finally, apply the algorithm from the corollary again. It is easy to see that the performed sequence of commands is valid and it has length $n(n-1) + n^2 + n(n-1) = n(3n-2)$, as desired.

1 point.

Second Solution. The starting lemma and the proof of the upper bound are the same as in the first solution and are worth the same number of points. In this solution we present a different way to obtain the lower bound on the answer.

Let L be the set of cups that visit the first position before the last position and similarly let R be the set of cups that visit the last position before the first position. Then C is the disjoint union of L and R, in particular $|L|+|R| = 2n$.

1 point.

Now consider two cups a and b such that a is to the left of b at the beginning. Then note that a and b have to be swapped at least once since otherwise a wouldn't visit the last position (and b wouldn't visit the first position). Moreover, if a and b are swapped exactly once, then note that we must have $a \in L$, $b \in R$.

2 points.

Hence, it follows that the total number of swaps is at least

$$\binom{2n}{2} \cdot 2 - |L| \cdot |R| \geqslant 2n(2n-1) - n^2 = n(3n-2),$$

where we used the AM-GM inequality to obtain $|L| \cdot |R| \leqslant \left(\frac{|L|+|R|}{2}\right) = n^2$.

2 points.

Notes on marking:

- Points obtained for different proofs of the lower bound are not additive, a student should be awarded the maximum of points obtained for a single approach.
- If a student states that a sequence of commands being valid is equivalent to each cup visiting each position, it should be awarded **0 points**. The reason for this is that this characterisation of valid sequences is trivial and not directly useful, whereas both solutions extensively make use of the characterisation presented in Lemma 1.