6^{TH} European Mathematical Cup

9^{th} December 2017–23^{rd} December 2017

Junior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problems and Solutions

Problem 1. Find all pairs (x, y) of integers that satisfy the equation

$$x^2y + y^2 = x^3.$$

(Daniel Paleka)

First Solution. Firstly, let us consider the case $x = 0$. In that case, it is trivial to see that $y = 0$, and thus we get the solution $(0, 0)$. From now on, we can assume $x \neq 0$.

1 point.

From the trivial $x^2|x^3$, the equation gives $x^2|x^2y + y^2 \Rightarrow x^2|y^2$, which means $x|y$.

1 point.

We use the substitution $y = kx$, where $k \in \mathbb{Z}$.

1 point.

The substitution gives us

$$kx^3 + k^2x^2 = x^3$$
$$kx + k^2 = x$$
$$k^2 = x(1 - k)$$

.

2 points.

Considering the greatest common divisor of k^2 and $1 - k$, we get

$$GCD(k^2, 1 - k) = GCD(k^2 + k(1 - k), 1 - k) = GCD(k, 1 - k) = GCD(k, 1 - k + k) = GCD(k, 1) = 1$$

3 points.

That leaves us with two possibilities.

a) $1 - k = 1 \Rightarrow k = 0 \Rightarrow x = 0$ which is not possible since $x \neq 0$.

1 point.

b) $1 - k = -1 \Rightarrow k = 2 \Rightarrow x = -4, y = -8$, which gives a solution to the original equation.

1 point.

Second Solution. We rearrange the equation into:

$$y^2 = x^2(x - y).$$

It can easily be shown that if $y \neq 0$, $x - y$ must be square.

1 point.

If $y = 0$, from the starting equation we infer $x = 0$, and we have a solution $(x, y) = (0, 0)$.
In the other case, we set $x = y + a^2$, where a is a positive integer. Taking the square root of the equation gives:

$$|y| = |x|a$$

.

1 point.

Because $x = y + a^2 > y$, it is impossible for y to be a positive integer, because then the equation would say $y = xa > x$, which is false. That means $y < 0$, and also:

$$-y = |x|a$$

2 points.

If x is positive, we can write:

$$-y = xa = (y + a^2)a = ay + a^3$$

which rearranges into

$$-y(a + 1) = a^3,$$

so a^3 is divisible by $a + 1$, which is not possible for positive a due to $a^3 = (a + 1)(a^2 - a + 1) - 1$.

2 points.

We see that x cannot be zero due to y being negative, so the only remaining option is that $x < 0$ also. We write:

$$-y = xa = -(y + a^2)a = -ay + a^3$$

which can similarly be rearranged into

$$-y(a - 1) = a^3,$$

and this time a^3 is divisible by $a - 1$.

1 point.

Analogously, we decompose $a^3 = (a - 1)(a^2 + a + 1) + 1$, so $a - 1$ divides 1 and the unique possibility is $a = 2$.

2 points.

The choice $a = 2$ gives $y = -8$ and $x = -4$, which is a solution to the original equation.

1 point.

Notes on marking:

- Points awarded for different solutions are not additive, a student should be awarded the maximal number of points he is awarded following only one of the schemes.
- Saying that $(0, 0)$ is a solution is worth **0 points**. The point is awarded only if the student argues that, disregarding the solution $(0, 0)$, we must only consider $x \neq 0$, or a similar proposition.
- Failing to check that $(0, 0)$ is a solution shall not be punished. Failing to check that $(-4, -8)$ is a solution shall result in the deduction of **1 point** only if a student did not use a chain of equivalences to discover the solution.

Problem 2. A regular hexagon in the plane is called *sweet* if its area is equal to 1. Is it possible to place 2000000 sweet hexagons in the plane such that the union of their interiors is a convex polygon of area at least 1900000?

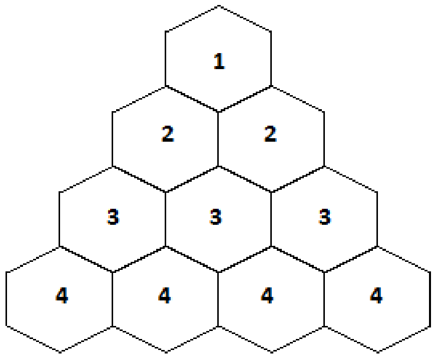
Remark: A subset S of the plane is called *convex* if for every pair of points in S, every point on the straight line segment that joins the pair of points also belongs to S. The hexagons may overlap.

(Josip Pupic, Borna Vukorepa)

Solution. It is possible to make such arrangement.

0 points.

We will stack hexagons in a triangular pattern shown below, where the first row has one hexagon, second row has two and so on. The pattern on the picture is a triangle with four rows.



3 points.

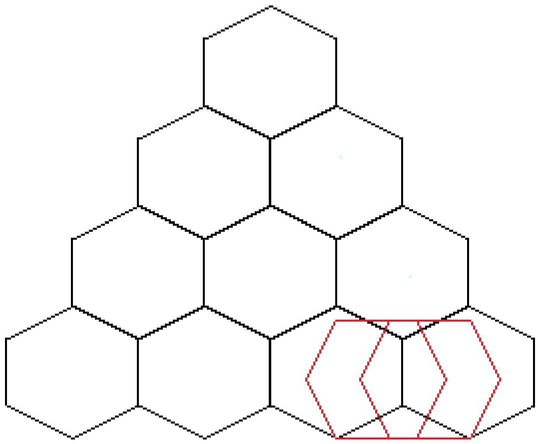
Such triangle with n rows has an area of $\frac{n(n+1)}{2}$ since that is the total number of hexagons used for that pattern.

1 point.

Setting $n = 1950$ gives us a triangle with 1950 rows. That figure has an area of 1902225 and the same number of hexagons is used. The only problem is that it is not convex.

1 point.

We can use the remaining hexagons to fix the non-convex parts of the figure, as shown below.



3 points.

Every non-convex part can be fixed with two hexagons, so in total we will need $1949 \cdot 3 \cdot 2 = 11694$ hexagons to make the figure convex. This is because there are 1949 non-convex parts on every side of our triangular pattern. Obviously, this is much less hexagons than we have remaining. The resulting figure is now convex, so this completes the proof.

2 points.

Notes on marking:

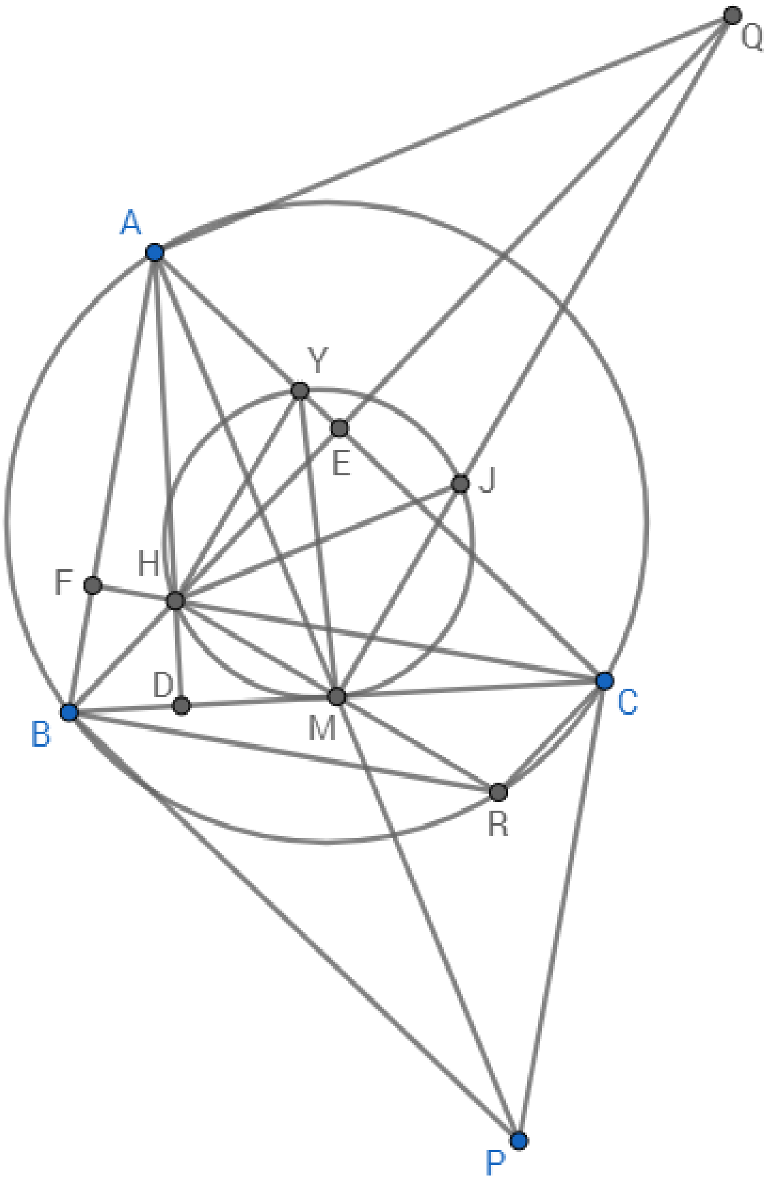
- Sketches of stacking the hexagons in any pattern will not be worth any points if there is no work done on them.

- No points are awarded for the claim that the construction is possible.
- There are many patterns for stacking the hexagons which can give the correct solution. Each of them should be marked the same way as this one.
- Work on patterns which can't produce the desired area will not be worth any points.

Problem 3. Let ABC be an acute triangle. Denote by H and M the orthocenter of ABC and the midpoint of side BC, respectively. Let Y be a point on AC such that YH is perpendicular to MH and let Q be a point on BH such that QA is perpendicular to AM. Let J be the second point of intersection of MQ and the circle with diameter MY. Prove that HJ is perpendicular to AM.

(Steve Dinh)

Solution. We present the following diagram:



0 points.

Since $\angle MHY = 90^\circ$, Y lies on the circle with diameter MY, so the quadrilateral $HMJY$ is cyclic.

1 point.

It follows that $\angle HJM = \angle HYM$. Since $QA \perp AM$,

$$HJ \perp AM \iff HJ \parallel QA \iff \angle HJM = \angle AQM \iff \angle HYM = \angle AQM.$$

Since $\angle YHM = \angle QAM = 90^\circ$, the latter is to equivalent to $\triangle AQM \sim \triangle HYM$.

1 point.

Now we have two different approaches to finish the solution:

First Approach (Synthetic). Let P, R be the reflections of A, H in M, respectively.
Then since $\angle YHM = \angle QAM = 90^\circ$, i.e. $\angle YHR = \angle QAP = 90^\circ$,

$$\triangle AQM \sim \triangle HYM \iff \frac{AQ}{HY} = \frac{AM}{HM} \iff \frac{AQ}{HY} = \frac{\frac{1}{2}AP}{\frac{1}{2}HR} \iff \frac{AQ}{HY} = \frac{AP}{HR} \iff \triangle AQP \sim \triangle HYR \iff \angle QPA = \angle YRH.$$

3 points.

Since M is the midpoint of BC, the quadrilaterals $ABPC$ and $HBRC$ are parallelograms.

1 point.

Since $CR \parallel HB$ and $HB \perp AC$, it follows that $\angle ACR = 90^\circ$. Hence $\angle YCR = \angle RHY = 90^\circ$, so the quadrilateral $YHRC$ is cyclic.

1 point.

It follows that $\angle YRH = \angle YCH = \angle ACF = 90^\circ - \angle BAC$.

1 point.

Since $BP \parallel AC$ and $AC \perp BQ$, we have $PBQ = 90^\circ$. Hence $\angle PBQ = \angle PAQ = 90^\circ$, so the quadrilateral $ABPQ$ is cyclic.

1 point.

It follows that $\angle QPA = \angle QBA = \angle EBA = 90^\circ - \angle BAC$.

1 point.

Finally, we conclude that $\angle YRH = \angle QPA$, as desired.

Second Approach (Trigonometric). We will show that $\triangle AQM \sim \triangle HYM$ by proving that

$$\frac{AQ}{AM} = \frac{HY}{HM}.$$

To begin with, let $a = BC, b = CA, c = AB$ and $\alpha = \angle BAC, \beta = \angle CBA, \gamma = \angle ACB$.

From right-angled $\triangle AEQ$ we get $AQ = \dfrac{AE}{\cos\angle EAQ} = \dfrac{AE}{\sin\angle MAC}$.

Then from right-angled $\triangle ABE$ we obtain $AE = AB\cos\angle BAE = c\cos\alpha$, so $AQ = \dfrac{c\cos\alpha}{\sin\angle MAC}$, i.e. $\dfrac{AQ}{AM} = \dfrac{c\cos\alpha}{AM\sin\angle MAC}$.

By the sine law applied to $\triangle AMC$, we obtain $\dfrac{AM}{\sin\angle ACM} = \dfrac{MC}{\sin\angle MAC}$, i.e. $AM\sin\angle MAC = \frac{a}{2}\sin\gamma$.

It follows that $\dfrac{AQ}{AM} = \dfrac{c\cos\alpha}{\frac{a}{2}\sin\gamma} = \dfrac{c}{\sin\gamma}\cdot\dfrac{2\cos\alpha}{a} = \dfrac{a}{\sin\alpha}\cdot\dfrac{2\cos\alpha}{a} = 2\cot\alpha$, where we used the sine law applied to $\triangle ABC$.

4 points.

To conclude, note that $\triangle AHY \sim \triangle BMH$ since $\angle HAY = \angle MBH = 90^\circ - \gamma$ and $\angle YHA = \angle HMB$ (angles with perpendicular rays). Then $\dfrac{HY}{HM} = \dfrac{AH}{BM} = 2\cot\alpha$, so we are done.

4 points.

Notes on marking:

- The points from different approaches are not additive, a student should be awarded the maximum of points obtained from one of them.

Problem 4. The real numbers x, y, z satisfy $x^2 + y^2 + z^2 = 3$. Prove the inequality

$$x^3 - (y^2 + yz + z^2)x + y^2z + yz^2 \leq 3\sqrt{3}$$

and find all triples (x, y, z) for which equality holds.

(Miroslav Marinov)

Solution. First let us notice the factorization of the left-hand side

$$x^3 - (y^2 + yz + z^2)x + y^2z + yz^2 = (x - y)(x - z)(x + y + z)$$

.

2 points.

Now we get the following inequalities

$$(x^3 - (y^2 + yz + z^2)x + y^2z + yz^2)^{\frac{2}{3}}$$
$$= \sqrt[3]{(x - y)^2(x - z)^2(x + y + z)^2}$$

1 point.

$$\overset{G-A}{\leqslant} \frac{1}{3}((x - y)^2 + (x - z)^2 + (x + y + z)^2)$$

3 points.

$$= \frac{1}{3}(3x^2 + 2y^2 + 2z^2 + 2yz)$$
$$= \frac{1}{3}(6 + x^2 + 2yz)$$
$$\overset{G-A}{\leqslant} \frac{1}{3}(6 + x^2 + y^2 + z^2)$$

1 point.

$$= \frac{9}{3} = 3$$

from where we get the required inequality by raising to the power of $\frac{3}{2}$.
In the case of equality, expressions $|x - y|, |x - z|$ and $|x + y + z|$ are all equal to $\sqrt{3}$ which we conclude from the first G-A inequality. From the case of equality in the second G-A inequality we conclude $y = z$. Now from $|x - y| = \sqrt{3}$ we get 2 cases:

a) $x - y = \sqrt{3} \Rightarrow |3y + \sqrt{3}| = \sqrt{3}$ from where we get $y = 0$ or $y = -\frac{2\sqrt{3}}{3}$ which gives us potential solutions $(\sqrt{3}, 0, 0)$ and $(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3})$. By checking only $(\sqrt{3}, 0, 0)$ remains.

b) $x - y = -\sqrt{3} \Rightarrow |3y - \sqrt{3}| = \sqrt{3}$ from where we get $y = 0$ or $y = \frac{2\sqrt{3}}{3}$ which gives us potential solutions $(-\sqrt{3}, 0, 0)$ and $(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3})$. By checking only $(-\frac{\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3})$ remains.

3 points.

Notes on marking:

- Factorization from the beginning can be spotted because y and z are obviously roots of the polynomial equation $x^3 - (y^2 + yz + z^2)x + y^2z + yz^2 = 0$ in the variable x.
- 1 point is to be deducted if potential solutions aren't checked out i.e. either $(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3})$ or $(-\sqrt{3}, 0, 0)$ is stated as solutions for the case of equality and.
- Proving the inequality is worth **7 points** while the rest is worth **3 points** non-depending on the way in which it was proved.