5^{TH} European Mathematical Cup

3^{rd} December 2016–11^{th} December 2016

Senior Category

Problems and Solutions

Problem 1. Is there a sequence $a_1, \ldots, a_{2016}$ of positive integers, such that every sum

$$a_r + a_{r+1} + \ldots + a_{s-1} + a_s$$

(with $1 \leqslant r \leqslant s \leqslant 2016$) is a composite number, but

a) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \ldots, 2015$;

b) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \ldots, 2015$ and $GCD(a_i, a_{i+2}) = 1$ for all $i = 1, 2, \ldots, 2014$?

$GCD(x, y)$ denotes the greatest common divisor of x, y.

(Matija Bucić)

First Solution. We will solve this problem for any length n of the sequence.

a) Yes, there is such sequence.

For this part, we will construct solution by taking n consecutive positive integers $a_i = m + i$, for some positive integer m. We will determine number m at the end of the proof.

Firstly, notice that two consecutive elements of the sequence are coprime, since they are consecutive numbers.

Every sum of consecutive members of sequence is of the form

$$(a+1) + (a+2) + \ldots + (b-1) + b = \frac{b(b+1)}{2} - \frac{a(a+1)}{2} = \frac{(b-a)(b+a+1)}{2}.$$

For $b \geqslant a + 3$, numerator of the expression above consists of two factors, each greater or equal to 3, and at least one of them is even, thus number is composite.

Thus, we have to choose m such that all sums of one and all sums of two consecutive members of sequence are composite. That is, following numbers need to be composite:

$$m+1, m+2, \ldots, m+n, 2m+3, 2m+5, \ldots 2m+(2n-1).$$

This is achieved for $m = (2n+1)! + 1$. Namely, numbers $(2n+1)! + k$ and $2 \cdot (2n+1)! + k$ are composite for all $2 \leqslant k \leqslant 2n+1$ since they are divisible by k, and greater than k.

b) Again, the answer is yes.

Similarly like in first part, we will take some n consecutive odd numbers: $a_i = 2m + (2i-1)$, for some positive integer m.

It is clear that they are integers, and they are positive.

We will have $GCD(a_i, a_{i+1}) = GCD(a_i, a_{i+2}) = 1$ because differences of mentioned numbers are always 2 or 4. Since numbers are odd, they have to be coprime.

Every sum of consecutive members of sequence is of the form

$$(2a+1) + (2a+3) + \ldots + (2b-3) + (2b-1) = b^2 - a^2 = (b-a)(b+a).$$

For $b \geqslant a + 2$ number from above is composite because both factors are greater or equal to 2.

Thus, we have to choose m such that all numbers

$$2m+1, 2m+3, \ldots, 2m+(2n-1)$$

are composite. This is achieved by taking $m = (2n)! + 1$, with similar arguments like in first part.

Second Solution. (For part b) only.)
We will show that there exists a sequence for b) part of the problem.
It is obvious that this will imply that the answer for the a) part of the solution is yes.
We will form the sequence by induction. For the basis, we will take $a_1 = 4$, $a_2 = 35$. Those numbers are composite, their sum is composite and they are coprime.
Let us assume that we have n positive integers with properties from the text of the problem. Let $p_1, p_2, \ldots, p_n, p_{n+1}$ be some prime numbers greater than

$$a_1 + a_2 + \ldots + a_n.$$

Notice that this immediately means that those primes are greater than any sum of consecutive numbers, and specially, that all those sums (including solely integers a_i) are coprime with mentioned primes.
We will get a_{n+1} by solving system of modular equations. Existence of such positive integer is provided by Chinese remainder theorem. The system is the following:

$$a_{n+1} \equiv 1\ (mod\ a_n,\ \ a_{n+1} \equiv 1\ (mod\ a_{n-1}),$$

$$a_{n+1} \equiv -(a_n + \ldots + a_{n-k})\ (mod\ p_k),\ \ k = 0, 1, 2, \ldots, n-1$$

$$a_{n+1} \equiv 0\ (mod\ p_{n+1}).$$

In first row we provided that $GCD(a_{n+1}, a_n) = GCD(a_{n+1}, a_{n-1}) = 1$.

In second two rows we provided that all sums of consecutive numbers including a_{n+1} are composite.

Chinese remainder theorem can be applied here since all primes are greater than a_n and a_{n-1}, and thus they are coprime.

Third Solution. (For part b) only.) As before, it is sufficient to prove the existence of the sequence for b) part only.
We will form recursion: $a_{-1} = 1, a_0 = 3, a_k = a_{k-1}^2 - a_{k-2}^2$, for $k \geqslant 1$. (Here values a_{-1} and a_0 are just auxiliary terms).
All numbers are positive integers, moreover we will prove that $a_k \geqslant a_{k-1} + 2$, which we get from induction:

$$a_k = a_{k-1}^2 - a_{k-2}^2 \geqslant a_{k-1}^2 - (a_{k-1} - 2)^2 = 4a_{k-1} - 4 \geqslant a_{k-1} + 2,$$

since $a_{k-1} \geqslant a_0 = 3$.
If there is some index k and some prime p such that p divides a_k and a_{k-1} or divides a_k and a_{k-2}, then from equation $a_k = a_{k-1}^2 - a_{k-2}^2$ we get that p divides a_{k-1} and a_{k-2}. In the same manner, p then divides a_{k-2} and a_{k-3}, it divides a_{k-3} and a_{k-4}, and so on, thus it divides a_{-1} and a_0, which is impossible.
Let us now prove that all sums of consecutive elements are composite:

$$a_r + \ldots + a_s = (a_{r-1}^2 - a_{r-2}^2) + \ldots + (a_{s-1}^2 - a_{s-2}^2) = a_{s-1}^2 - a_{r-2}^2 = (a_{s-1} - a_{r-2})(a_{s-1} + a_{r-2}).$$

First factor is greater than 1 since $a_{s-1} \geqslant a_{r-1} \geqslant a_{r-2} + 2$. Second factor is clearly greater than 1, hence the product is composite.

Fourth Solution. (For part a) only.)
The answer is yes.
Similarly like in the first solution, we will take sequence of consecutive third powers of positive integers: $a_i = (i+1)^3$.
Like in first solution, consecutive elements are coprime. It is clear that all numbers are positive integers.
All possible sums of consecutive elements are of the form

$$(a+1)^3 + (a+2)^3 + \ldots + (b-1)^3 + b^3 = \left(\frac{b(b+1)}{2}\right)^2 - \left(\frac{a(a+1)}{2}\right)^2 = \left(\frac{b(b+1)}{2} - \frac{a(a+1)}{2}\right)\left(\frac{b(b+1)}{2} + \frac{a(a+1)}{2}\right) =$$

$$= ((a+1) + (a+2) + \ldots + (b-1) + b)\left(\frac{b(b+1)}{2} + \frac{a(a+1)}{2}\right).$$

Second factor is greater or equal than first one. Second is greater than 1 if all elements of sequence are greater than 1. Since we chose numbers in that way, the number is composite.

Problem 2. For two positive integers a and b, Ivica and Marica play the following game: Given two piles of a and b cookies, on each turn a player takes $2n$ cookies from one of the piles, of which he eats n and puts n of them on the other pile. Number n is arbitrary in every move. Players take turns alternatively, with Ivica going first. The player who cannot make a move, loses. Assuming both players play perfectly, determine all pairs of numbers (a, b) for which Marica has a winning strategy.

(Petar Orlić)

Solution. Marica wins the game if $|a-b| \leqslant 1$, otherwise Ivica wins.

We will say that a player is in a losing position if it is his turn and $|a-b| \leqslant 1$, while calling all other positions winning positions. It is easy to see that the only positions in which one cannot make a move are $(0,0)$,$(0,1)$,$(1,0)$,$(1,1)$ and that they are all losing positions.

Claim 1. *If a player is in a losing position, then regardless of his move he must leave a winning position for the other player.*

Proof. If the piles are of sizes x and $x+1$, then after a move they will have sizes $x-2k$ i $x+k+1$ (their difference is $3k+1$) or $x+k$ i $x-2k+1$ (their difference is $3k-1$). In both cases, the difference is at least 2. If the piles have x and x cookies each, then after a move they will have $x-2k$ and $x+k$ cookies (there difference is $3k$, which is at least 3). Since the difference of the number of cookies is always bigger than 1, we have proven that this is a winning position. □

Claim 2. *A player who is in a winning position can always leave a losing position after his turn.*

Proof. If the piles are of sizes x and $x+3a$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and and leave two piles containing $x+a$ and $x+a$ cookies. If the piles are of sizes x and $x+3a+1$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and leave two piles containing $x+a$ and $x+a+1$ cookies.

If the piles are of sizes x and $x+3a-1$ (where $a \geqslant 0$), one can take $2a$ cookies from the second pile and and leave two piles containing $x+a$ and $x+a-1$ cookies. Since the difference in each case is less than 2, thus a player can always leave a loosing position if he is in a winning position. □

We have now proven that if Ivica is in a loosing position in the begging, Marica can always ensure that he is in a winning position and win. Similarly, if Ivica is in a winning position in the begging, he can always ensure that he is in a winning position and win. So, Marica wins only when Ivica is in a losing position in his first turn. This is true only when $|a-b| \leqslant 1$.

Problem 3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that equality

$$f(x + y + yf(x)) = f(x) + f(y) + xf(y)$$

holds for all real numbers x, y.

(Athanasios Kontogeorgis)

Solution. We easily see that $f(x) = 0, x \in \mathbb{R}$ and $f(x) = x, x \in \mathbb{R}$ are solutions. Let us assume that f satisfies the given equation but is not a constant or identity.

Throughout the proof, we denote by (x_0, y_0) the initial equation with $x = x_0, y = y_0$.

Then, $(-1, y)$ implies $f(-1 + y(1 + f(-1))) = f(-1)$. Let us assume that $1 + f(-1) = c \neq 0$. Then, for any $cy - 1$ achieves all real numbers and hence $f(z) = f(-1)\ \forall z \in \mathbb{R}$ so f is a constant, a contradiction. Hence, $f(-1) = -1$.

Let us assume that there is some $\alpha \in \mathbb{R}$ such that $f(\alpha) = -1$, but $\alpha \neq -1$. Then, $(\alpha, y) : f(\alpha) = f(\alpha) + f(y) + \alpha f(y) \implies 0 = f(y)(1 + \alpha)$. Since $\alpha \neq -1$, we get $f(y) = 0, y \in \mathbb{R}$, a contradiction. Thus, we have shown

$$f(x) = -1 \iff x = -1 \tag{1}$$

$$(x, -1) : \ f(x - 1 - f(x)) = f(x) - 1 - x. \tag{2}$$

Since we assumed that f is not the identity, there exists a real number x_0 such that $f(x_0) \neq x_0$. We set $a := f(x_0) - x_0 \neq 0$. Putting $x = x_0$ in the above equation gives:

$$f(-1 - a) = a - 1. \tag{3}$$

We get from $(-1 - a, y)$: and equation (3)

$$f(-1 - a + ya) = a - 1 - af(y). \tag{4}$$

If we now put $y = 1$, we get $a(1 - f(1)) = 0$ so as $a \neq 0$ we get $f(1) = 1$.

Now $(1, 1)$ gives us $f(3) = 3$.

Putting $(1, y - 1)$ gives us

$$f(2y - 1) = 2f(y - 1) + 1. \tag{5}$$

Using $f(3) = 3$ in (3) with $y = 3$ we get $f(2a - 1) = -2a - 1$, while using $y = a$ in (5) we get $f(2a - 1) = 2f(a - 1) + 1$, combining the two gives us

$$f(a - 1) = -1 - a. \tag{6}$$

We get from $(a - 1, 2 - y)$:

$$f(-a - 1 + ay) = -1 - a + af(2 - y). \tag{7}$$

Combining this with (4) we get:

$$a(f(y) + f(2 - y) - 2) = 0. \tag{8}$$

So as $a \neq 0$, we get $f(y) + f(2 - y) = 2$ for all y.

Putting $y = 1 + 2x$ here, gives $f(1 + 2x) + f(1 - 2x) = 2$, which when combined with 5 with $y = x + 1$ gives, $f(1 - 2x) = 1 - 2f(x)$.

While (5) for $y = 1 - x$ gives $f(1 - 2y) = 1 + 2f(-y)$, which combined with the above implies $f(-x) = -f(x)$ for all x.

Let us put $(x, -y)$ in initial equation, and then subtract the original equation (for (x, y)). We obtain:

$$f(x + y(1 + f(x))) + f(x - y(1 + f(x))) = 2f(x). \tag{9}$$

We substitute y with

$$\frac{y}{1 + f(x)}$$

and get

$$f(x + y) + f(x - y) = 2f(x), \tag{10}$$

which is valid for all x, y, with $f(x) + 1 \neq 0 \iff x \neq -1$. But, from f being odd and (8), we see that this is valid for $x = -1$, as well. In (10) we put $x = y$ to obtain $f(2x) = 2f(x)$. In the same equation we put $\frac{x-y}{2}$, $\frac{x+y}{2}$ and obtain

$$f(x) + f(y) = f(x + y). \tag{11}$$

Using this additivity, we can simplify the original equation:

$$f(xf(y)) = yf(x) \tag{12}$$

In the last equation we can firstly put $(1, y) \implies f(f(y)) = y$ and secondly $f(y)$ instead of y: $f(xy) = f(x)f(y)$.

It is well known that from identities $f(1) = 1$, $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$ we can conclude that $f(x) = x$. Which is a contradiction.

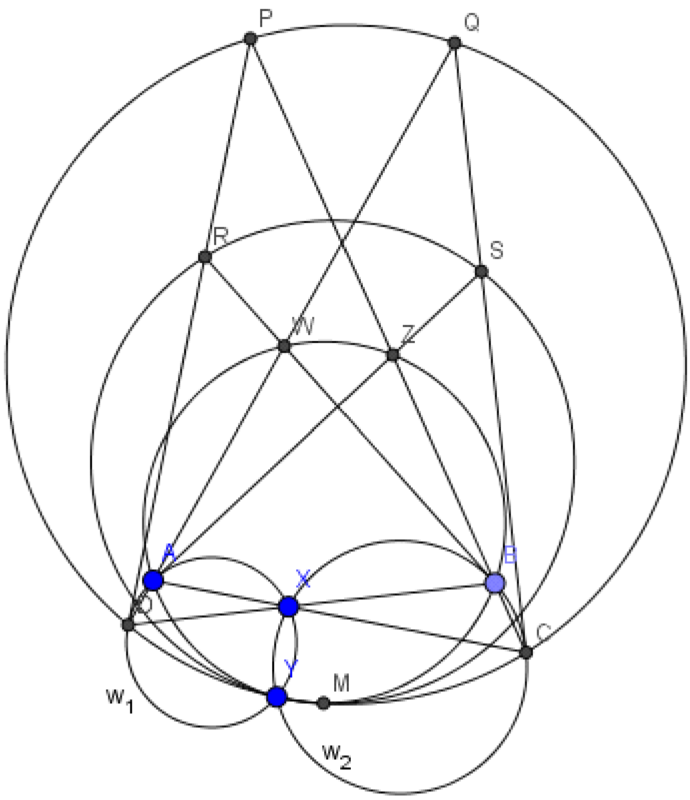
For the well known claim, we notice that $f(x^2) = f(x)^2$ implies $f(x) \geq 0$ for $x \geq 0$, which implies, combined with $f(x+y) = f(x)+f(y)$ that f is non-decreasing which in turn is enough to combine with the standard density of rationals argument to solve Cauchy's equation.

Hence, the functions presented at the start give all possible solutions.

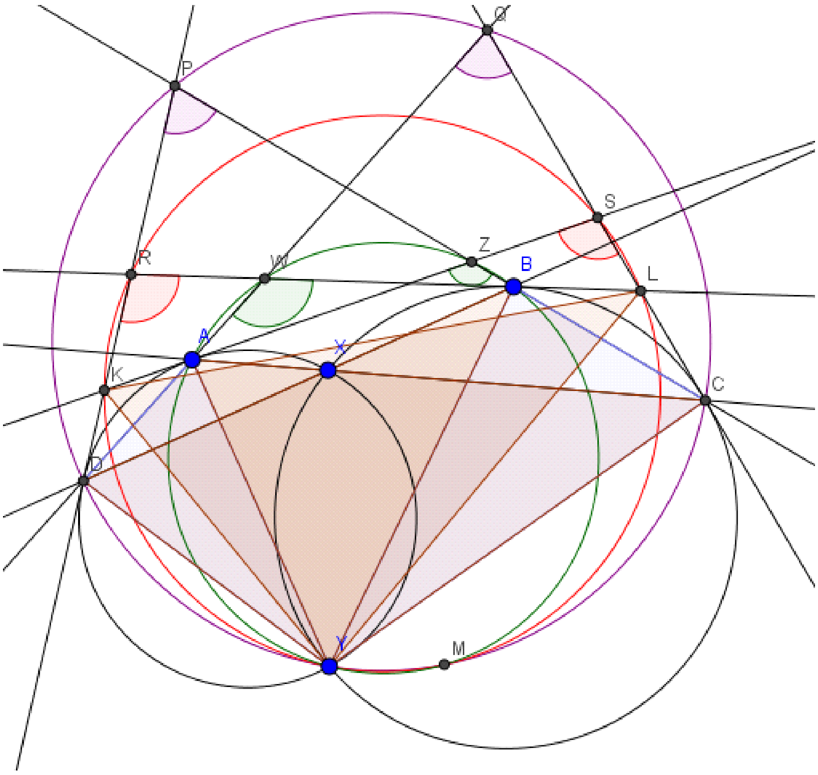
Problem 4. Let C_1, C_2 be circles intersecting in X, Y. Let A, D be points on C_1 and B, C on C_2 such that A, X, C are collinear and D, X, B are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in P, R respectively. The tangent to C_2 at C intersects AD and tangent to C_1 at A, in Q, S respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles YWZ, RSY and PQY have two points in common, or are tangent in the same point.

(Misiakos Panagiotis)

Solution. We present the following sketch:



Consider K, L the intersections of the pairs of tangents at (A, D) to C_1 and (B, C) to C_2 respectively.



Notice that $\angle WAZ = \angle KAD = \angle AXD = \angle BXC = \angle LBC = \angle ZBW$. So $WZBA$ is a cyclic quadrilateral. Furthermore, $\angle AZB = \angle AZC = 180^\circ - \angle ZAC - \angle ZCA = 180^\circ - \angle AYX - \angle BYX = 180^\circ - \angle AYB$. Thus the circle from A, Y, B passes from Z, and since W, Z, B, A are concyclic W, Z, B, Y, A belong to the same circle.

Analogous angle chase gives P, Q, C, Y, D concyclic.

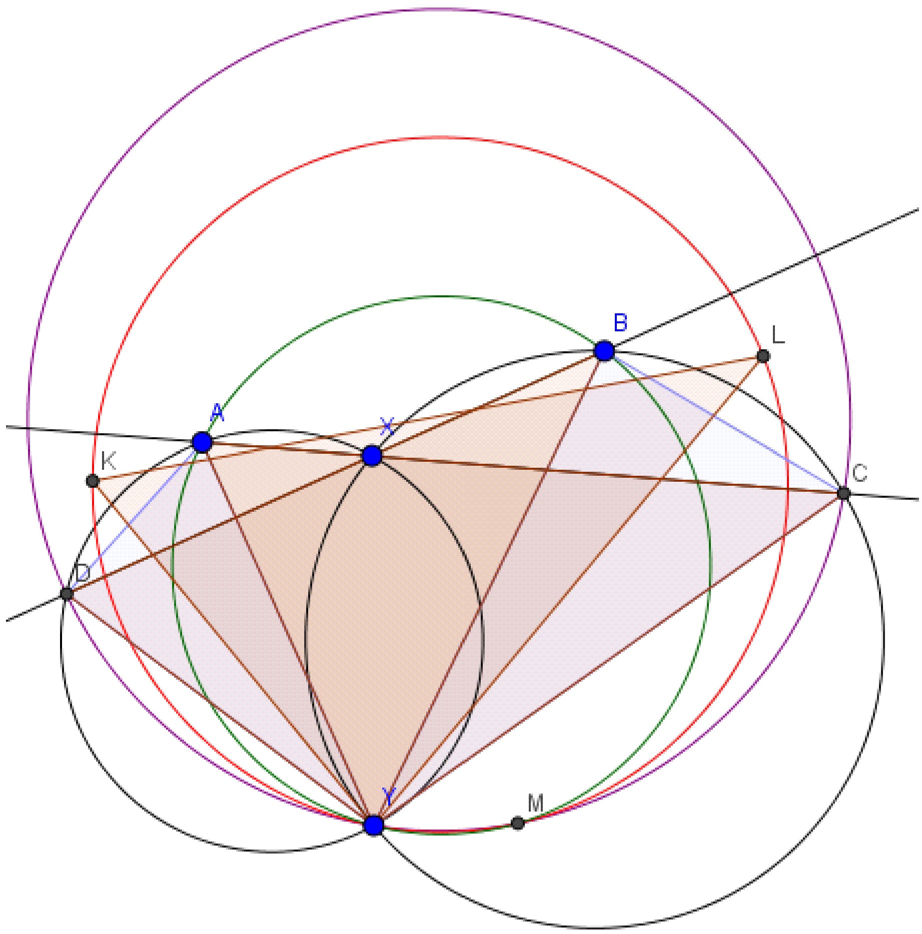
K, Y, L, S, R are also concyclic, this follows from $\angle ASC = 180^\circ - \angle SAC - \angle SCA = 180^\circ - \angle AYC$.

We have, $\angle XDY = \angle XAY$ and $\angle YBX = \angle YCX$ which implies $\triangle DYB \sim \triangle YAC$. This implies $\angle DYB = \angle AYC$.

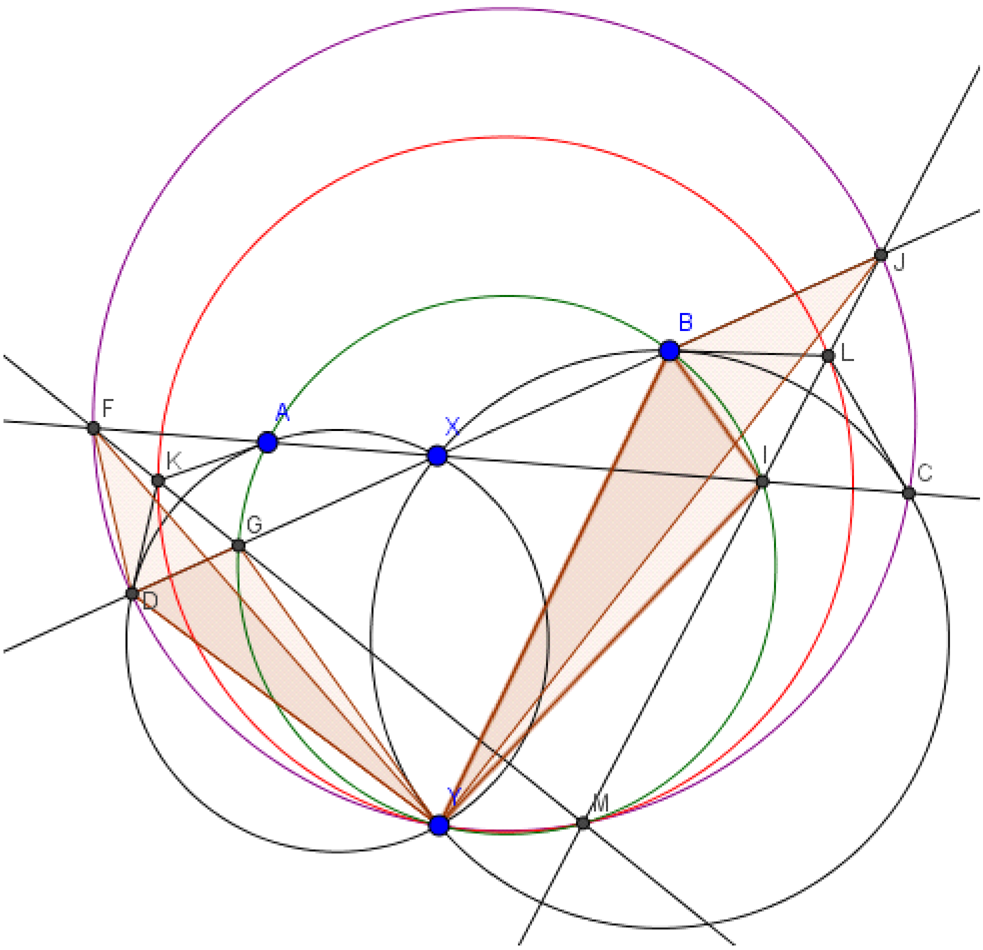
We have $\angle DRB = 180^\circ - \angle RDB - \angle DBR = 180^\circ - \angle DYX - \angle XYB = 180^\circ - \angle DYB = 180^\circ - \angle AYC = 180^\circ - \angle AYX - \angle XYC = 180^\circ - \angle XCS - \angle XAS = \angle ASC$. This implies $KRSL$ is cyclic.

$\triangle DYB \sim \triangle YAC$ also implies $\angle DYA = \angle BYC$, as well as $\frac{DY}{AY} = \frac{YB}{YC}$ which implies $\triangle DYA \sim \triangle BYC$. This further implies the isosceles triangles AKD and LBC have same angles so quadrilaterals $DYAK$ and $BYCL$ are also similar, in particular implying $\angle KYD = \angle LYB$. This in turn implies $180^\circ - \angle DRL = \angle DYB = \angle KYL$ which in turn implies Y is on the same circle as K, R, S, L.

We now proceed to show circumcircles of YKL, YDC, YAB have two common points.



Let F, J be the points of intersections of AC, BD with circle DYC respectively and G, I be the points of intersection of BD,AC with circle ABY.



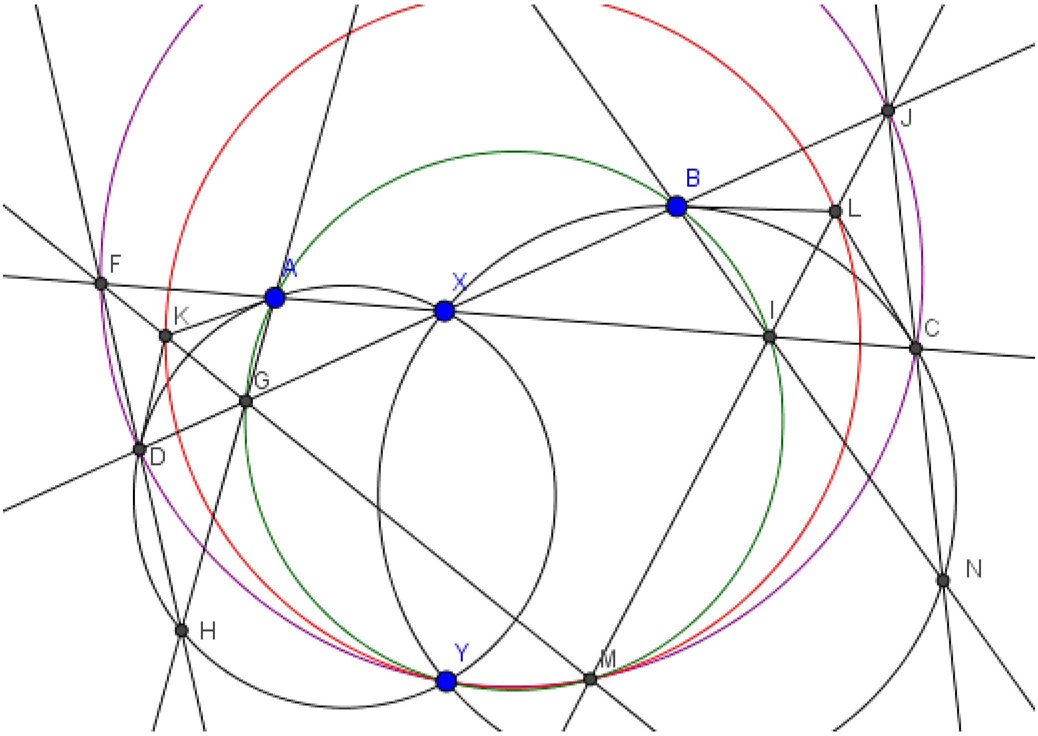
Now let M be the intersection of lines FG, JI. We will eventually prove that this will be a second common point for the three circles.

First we show that the FDY, BJY are similar. For this note that $\angle FXJ = \angle XJC + \angle XCJ = \angle FYJ + \angle DJC = \angle FYJ + 180^\circ - \angle DYC \implies \angle FYJ = \angle FXJ - 180^\circ + \angle DYC = \angle DYC - \angle AXD = \angle AYC = \angle DYB$. Thus $\angle FYJ = \angle DYB$ and so $\angle FYD \sim \angle JYB$. While $\angle DJY = \angle DFY$ showing $\triangle FDY \sim JBY$ as claimed.

Also $\triangle DGY \sim \triangle BIY$ are similar , since $\angle AXB = \angle XBI + \angle BIX = \angle XBI + \angle AYB = 180^\circ - \angle GYI + \angle AYB = 180^\circ - \angle AYG - \angle BYI \implies \angle AXD = \angle AYG + \angle IYB \implies \angle AYG + \angle GYD = \angle AYG + \angle IYB \implies \angle GYD = \angle IYB$. Also, $\angle BIY = \angle DGY$, $\angle YBI = \angle YAI = \angle YAX = \angle YDX = \angle YDG$ and we get our result.

Now we get that the spiral similarity that sends $D \to B$ and $F \to J$ also sends $G \to I$, so $\triangle FGY \sim \triangle JIY$, so $\angle YGM = \angle YIM$ and $\angle YFM = \angle YJM$, so M belongs to both of the circumcircles of FYJ and GIY, hence M is the (other than Y) common point of circumcircles of ABY and CDY.

Since $\angle FMJ = \angle FYJ = \angle DYB = \angle KYL$ it remains to show that K, L belong on the Lines FG, JI respectively (then circle KYL would pass through M.)



Let H denote the point of intersection of lines AG, FD. Then $\angle HDX = 180^\circ - \angle FDX = 180^\circ - \angle FMJ = \angle GAI = \angle GAX$, so H belongs to the circumcircle of triangle ADX.

Similarly denote N (the intersection of lines BI, JC) and it will for analogous reasons belong to the circumcircle of $\triangle BXC$.

Now from Pascal's theorem for the hexagons $AAXDDH$ and $BBXCCN$ we derive that F, K, G as well as J, L, I are collinear. The conclusion follows.