5$^{\text{TH}}$ European Mathematical Cup

3rd December 2016–11th December 2016

Senior Category

Problem 1. Is there a sequence $a_1, \dots, a_{2016}$ of positive integers, such that every sum

$$a_r + a_{r+1} + \ldots + a_{s-1} + a_s$$

(with $1 \leqslant r \leqslant s \leqslant 2016$) is a composite number, but

a) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \dots, 2015$;

b) $GCD(a_i, a_{i+1}) = 1$ for all $i = 1, 2, \dots, 2015$ and $GCD(a_i, a_{i+2}) = 1$ for all $i = 1, 2, \dots, 2014$?

$GCD(x, y)$ denotes the greatest common divisor of x, y.

(Matija Bucić)

Problem 2. For two positive integers a and b, Ivica and Marica play the following game: Given two piles of a and b cookies, on each turn a player takes $2n$ cookies from one of the piles, of which he eats n and puts n of them on the other pile. Number n is arbitrary in every move. Players take turns alternatively, with Ivica going first. The player who cannot make a move, loses. Assuming both players play perfectly, determine all pairs of numbers (a, b) for which Marica has a winning strategy.

(Petar Orlić)

Problem 3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that equality

$$f(x + y + yf(x)) = f(x) + f(y) + xf(y)$$

holds for all real numbers x, y.

(Athanasios Kontogeorgis)

Problem 4. Let C_1, C_2 be circles intersecting in X, Y. Let A, D be points on C_1 and B, C on C_2 such that A, X, C are collinear and D, X, B are collinear. The tangent to circle C_1 at D intersects BC and the tangent to C_2 at B in P, R respectively. The tangent to C_2 at C intersects AD and tangent to C_1 at A, in Q, S respectively. Let W be the intersection of AD with the tangent to C_2 at B and Z the intersection of BC with the tangent to C_1 at A. Prove that the circumcircles of triangles YWZ, RSY and PQY have two points in common, or are tangent in the same point.

(Misiakos Panagiotis)