4TH EUROPEAN MATHEMATICAL CUP

5th December 2015–13th December 2015

Junior Category

Problems and Solutions

Problem 1. We are given an $n \times n$ board. Rows are labeled with numbers 1 to n downwards and columns are labeled with numbers 1 to n from left to right. On each field of the board we write the number $x^2 + y^2$ where (x, y) are its coordinates. We are given a figure and can initially place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by n,
- those fields are point reflected with respect to the center of the board.

Can all the fields be visited in case:

a) $n = 4$,

b) $n = 5$?

(Josip Pupić)

Solution. a) The answer is NO.

	1	2	3	4
1	2	5	10	17
2	5	8	13	20
3	10	13	18	25
4	17	20	25	36

	1	2	3	4
1	2	1	2	1
2	1	0	1	0
3	2	1	2	1
4	1	0	1	0

On the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We will denote *field* i for a field with number i written on it in the right table. Let's assume that we can visit all of the fields. That means that at some point we will visit a *field* 1. Obviously, when using the first type of move, we can visit any other *field* 1 which hasn't yet been visited. Also, it easy to notice, that for *field* 1, the reflection of that field is also a *field* 1. That means that both types of moves lead to another *field* 1. Also, in same fashion we conclude that for the each step, if the figure is on the *field* 1, then in the step after (if that wasn't the last one) and in the step before (if that wasn't the first one) should be *field* 1.

Now we conclude that the first visited *field* 1 must be the field visited in the first step. Same way we conclude that the last visited *field* 1 must be the field visited in the last step. But, we know that all of *fields* 1 are visited consecutively, in exactly 8 moves (because there are 8 *fields* 1), while there are exactly 16 moves that we have to make. This leads to contradiction.

b) The answer is YES.

	1	2	3	4	5
1	2	5	10	17	26
2	5	8	13	20	29
3	10	13	18	25	34
4	17	20	25	36	41
5	26	29	34	41	50

	1	2	3	4	5
1	2	0	0	2	1
2	0	3	3	0	4
3	0	3	3	0	4
4	2	0	0	2	1
5	1	4	4	1	0

Again, on the left we have the board from the problem, on the right we have the same board, but with remainders of the values from the board instead of the values themselves.

We can move from any field to another with the same number written on the field in the right table by using the second move.

One idea to visit all the fields is the following:

- Find the 4 pairs of the fields of types *field* i and *field* j, such that all 8 fields are different, in each pair $i \neq j$, those two field in one pair are symmetric, and the second member of the n-th pair has the same value on the right board as the first member of the $(n+1)$-th pair. Also, we want that all the values of the right table are mentioned through members of those pairs. For example:

$$((2,2),(4,4)),\ ((1,4),(5,2)),\ ((3,5),(3,1)),\ ((2,1),(4,5))$$

- Now, the algorithm is: after second member of n-th pair and before the first member of the $(n+1)$-th pair visit all fields by using the first step. Of course, before first pair and after fourth pair move in similar way. Jump from the first member of the pair to the second member of the pair by using second step.

This is one of the ways to do it: We start with the field $(3,3)$. Then we visit all of the *field*s 3, using the first move, in any way as long as the last visited field is $(2,2)$. Then, using the second move, we visit the field $(4,4)$. Again, using the first move we visit all *field*s 2 in any way as long as the last visited field is $(1,4)$. Using the second move we visit the field $(5,2)$. Then, using the first move we visit all *field*s 4 in any way as long as the last visited field is $(3,5)$. In same fashion, using the second move we visit the field $(3,1)$. After visiting all *field*s 0 in any way as long as the last visited field is $(2,1)$, we visit the field $(4,5)$ using the second move. We conclude by visiting all *field*s 1 in any way.

Problem 2. Let m, n, p be fixed positive real numbers which satisfy $mnp = 8$. Depending on these constants, find the minimum of

$$x^2 + y^2 + z^2 + mxy + nxz + pyz,$$

where x, y, z are arbitrary positive real numbers satisfying $xyz = 8$. When is the equality attained? Solve the problem for:

a) $m = n = p = 2$,

b) arbitrary (but fixed) positive real m, n, p.

(Stijn Cambie)

First Solution. a) Use AM-GM and $xyz = 8$ to get

$$x^2 + y^2 + z^2 + xy + xy + yz + yz + xz + xz \geqslant 9\sqrt[9]{x^6y^6z^6} = 36.$$

We have equality for $x = y = z = 2$.

b) Using $xyz = 8$, we can transform the given expression:

$$x^2 + y^2 + z^2 + mxy + nxz + pyz = x^2 + \frac{8p}{x} + y^2 + \frac{8n}{y} + z^2 + \frac{8m}{z}$$

Since all numbers are positive reals, we can apply AM-GM inequality to get:

$$x^2 + \frac{8p}{x} = x^2 + \frac{4p}{x} + \frac{4p}{x} \geqslant 6\sqrt[3]{2p^2}$$

When we apply the same procedure for x, y, z and sum the inequalities, we get:

$$x^2 + y^2 + z^2 + mxy + nxz + pyz = x^2 + \frac{8p}{x} + y^2 + \frac{8n}{y} + z^2 + \frac{8m}{z} \geqslant 6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}).$$

In order to get equality, we must have equality in all above inequalities and that happens for

$$x = \sqrt[3]{4p},$$
$$y = \sqrt[3]{4n},$$
$$z = \sqrt[3]{4m}.$$

Desired minimum is therefore

$$6\sqrt[3]{2}(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}).$$

Second Solution. We only present solution for *b*) part here, marking scheme for *a*) part is the same as in first solution. We use weighted AM-GM:

$$x^2 + y^2 + z^2 + mxy + nxz + pyz =$$

$$\sqrt[3]{p^2}\frac{x^2}{\sqrt[3]{p^2}} + \sqrt[3]{n^2}\frac{y^2}{\sqrt[3]{n^2}} + \sqrt[3]{m^2}\frac{z^2}{\sqrt[3]{m^2}} + 2\sqrt[3]{m^2}\frac{mxy}{2\sqrt[3]{m^2}} + 2\sqrt[3]{n^2}\frac{nxz}{2\sqrt[3]{n^2}} + 2\sqrt[3]{p^2}\frac{pyz}{2\sqrt[3]{p^2}} \geq$$

$$3\left(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}\right) \cdot \sqrt[3\left(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2}\right)]{\left(\frac{x^2}{\sqrt[3]{p^2}}\right)^{\sqrt[3]{p^2}} \left(\frac{y^2}{\sqrt[3]{n^2}}\right)^{\sqrt[3]{n^2}} \left(\frac{z^2}{\sqrt[3]{m^2}}\right)^{\sqrt[3]{m^2}}}\cdot$$

$$\sqrt[3\left(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2}\right)]{\left(\frac{\sqrt[3]{m}xy}{2}\right)^{2\sqrt[3]{m^2}} \left(\frac{\sqrt[3]{n}xz}{2}\right)^{2\sqrt[3]{n^2}} \left(\frac{\sqrt[3]{p}yz}{2}\right)^{2\sqrt[3]{p^2}}}$$

$$= 3\left(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}\right) \cdot \sqrt[3\left(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2}\right)]{\left(\frac{xyz}{2}\right)^{2(\sqrt[3]{m^2}+\sqrt[3]{n^2}+\sqrt[3]{p^2})}}$$

$$= 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt[3]{\left(\frac{xyz}{2}\right)^2} = 3(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}) \cdot \sqrt[3]{4^2} = 6\sqrt[3]{2}\left(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}\right)$$

We have shown that the minimum value the expression can take is $6\sqrt[3]{2}\left(\sqrt[3]{m^2} + \sqrt[3]{n^2} + \sqrt[3]{p^2}\right)$. Equality can only be achieved when $x = \sqrt[3]{4p}, y = \sqrt[3]{4n}, z = \sqrt[3]{4m}$.

Problem 3. Let $d(n)$ denote the number of positive divisors of n. For positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \ldots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almost perfect numbers.

(Paulius Ašvydis)

First Solution. Alternative way to define $f(n)$ is

$$f(n) = \sum_{k|n, k\geq 1} d(k).$$

Let $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ be the prime factorisation of n. We have $d(n) = \prod_{i=1}^{r}(a_i + 1)$.

We prove the function f is multiplicative, in particular, given coprime n, m we have $f(nm) = f(n)f(m)$.

Using n, m are coprime for the second inequality and the fact that function d is multiplicative we get:

$$f(nm) = \sum_{k|nm} d(k) = \sum_{k_1|n, k_2|m} d(k_1k_2) = \sum_{k_1|n, k_2|m} d(k_1)d(k_2) = \left(\sum_{k_1|n} d(k_1)\right)\left(\sum_{k_2|n} d(k_2)\right) = f(n)f(m)$$

If $r = 1$ we have $n = p_1^{a_1}$. We note that divisors of n are $1, p_1, p_1^2, \cdots, p_1^{a_1}$ so $f(n) = \sum_{i=0}^{a_1}(i + 1) = \dfrac{(a_1 + 1)(a_1 + 2)}{2}$.

Combining this with the multiplicativity result for f we deduce $f(n) = \prod_{i=1}^{r} \dfrac{(a_i + 1)(a_i + 2)}{2}$.

We now prove that for primes $p \geq 5$ and $p = 3$ provided $a \geq 3$ we have $f(p^a) = \frac{(a+1)(a+2)}{2} < \frac{2}{3}p^a$ by induction on a. As a basis $3 < \frac{2p}{3}$ for $p \geq 5$ and $6 < \frac{2}{3} \cdot 3^3$. For the step it is enough to notice that $\frac{a+3}{a+1} \leq 2 < p$ in both cases.

Similarly we can prove for $p = 2$ that $f(p^a) < p^a$ provided $a \geq 4$. By explicitly checking the remaining cases $p = 2$ and $a = 1, 2, 3$ and $p = 3$ $a = 1, 2$ we conclude $f(p^a) \leq \frac{3}{2}p^a$ for all p, a and $f(p^a) \leq p^a$ for all $p \geq 3$ and $p = 2, a \geq 4$.

Assuming $f(n) = n$ we would have $\prod_{i=1}^{k} \dfrac{f(p_i^{a_i})}{p^{a_i}} = 1$ so the above considerations imply that only possible prime divisors are $2, 3$. If $k = 1$ the only possible solution is $n = 3$. If $k = 2$ we have $p_1 = 2, p_2 = 3$ and $1 \leq a_1 \leq 2$ and $1 \leq a_2 \leq 2$ which give 4 cases to check giving the other 2 solutions $n = 18, 36$.

So, all almost perfect numbers are $3, 18, 36$.

Second Solution. We hereby present one similar but different solution which does not use a lot of properties of the function f.

Firstly, we will prove the following lemma:

Lemma: For any positive integer $n > 1$ and prime p we have

$$f(pn) \leqslant 3f(n).$$

The equality holds if and only if $GCD(p, n) = 1$. *Proof:* For every integer m we have that the set of divisors of the number pm is the union of the following two sets:

- set of divisors of m,
- set of divisors of m multiplied by p.

Also, those two mentioned sets are disjoint if and only if $GCD(p, m) = 1$ (if we have that p, m are disjoint, then it is obvious that none of the divisors of pm are in both sets; if they are not coprime, then the number p belongs to both sets).

This is why we have $d(pm) \leqslant 2d(m)$ and

$$f(pn) = \sum_{k|pn} d(k) \leqslant \sum_{k|n} d(k) + \sum_{k|n} d(pk) \leqslant f(n) + \sum_{k|n} 2d(k) = 3f(n).$$

In both inequalities equality holds if and only if sets from before are disjoint, i.e. when $GCD(p, n) = 1$.

Also, we simply see that $f(2^k) = d(1) + d(2) + \ldots + d(2^k) = 1 + 2 + \ldots + (k + 1) = \frac{(k+1)(k+2)}{2}$.

Notice that if for some positive integer n we have $f(n) < n$, then for every $p \geqslant 3$ we have $f(pn) \leqslant 3f(n) \leqslant pf(n) < pn$. Consequently, if $f(n) < n$, then for every odd m we have $f(mn) < mn$. Because of this, we will introduce new terms. Number n is *nice multiple* of m if $m \mid n$ and $\frac{m}{n}$ is odd number. Analogously, we define *nice divisor*. Our statement from above is: if for some n we have $f(n) < n$, then neither of its nice multiples is almost perfect number. Our strategy will be the following: check the cases of the "small" numbers and see ratio of numbers n and $f(n)$. When we have that $n > f(n)$, conclude that there are not almost perfect numbers among their nice multiples. With formula for $f(2^k)$ conclude that for sufficiently big k (when $f(2^k) < 2^k$) this is enough to conclude that there are no more almost perfect numbers. By induction, it is simple to prove that $f(2^k) < 2^k$ for $k \geqslant 4$. Thus, there are no almost perfect numbers of the form $2^k \cdot m$, where $k \geqslant 4$ and m is odd, since they all have 2^k as their nice divisor. We only have to check the numbers of the form $2^k \cdot m$, where where $k \leqslant 3$ and m is odd.

First case: $k = 0$

For any odd prime p we have $f(p) = d(1) + d(p) = 3 \leqslant p$. From that we see that $n = 3$ is solution. Moreover, we do not have any more solutions: if some odd number has a prime divisor different from 3, since $f(p) < p$ this number can not be almost perfect number; if it is a power of 3 bigger than 3, since $f(9) < 3f(3) = 9$, there are no more solutions as well (9 is nice divisor of every power of 3 bigger that 3).

Second case: $k = 1$

For any odd prime we have $f(2p) = 3f(2) = 9$. If $p > 5$ then we have $2p > f(2p)$, so for all almost perfect numbers of the form $2^1 \cdot m$ number m has to have prime divisors 3 and/or 5.

We directly see that neither 6 or 10 is almost perfect. So, in this case, almost perfect number has to have a nice divisor of the form $2 \cdot 9$, $2 \cdot 15$ or $2 \cdot 25$. For $n = 18$ we have another solution, in other two cases we have inequality $f(n) < n$. If we want to seek new solution in this case, since they cannot be nice multiples of 30 and 50, the only possibility is that almost perfect number has nice divisor $2 \cdot 27$. But we have (equality case in lemma) that $f(2 \cdot 27) < 3f(2 \cdot 9) = 2 \cdot 27$. So, there are no more solutions in this case.

Third case: $k = 2$

For any odd prime we have $f(4p) = 3f(4) = 18$. If $p > 5$ then we have $4p > f(4p)$, so for all almost perfect numbers of the form $2^1 \cdot m$ number m has to have prime divisors 3 and/or 5.

We directly see that neither 12 or 20 is almost perfect. So, in this case, almost perfect number has to have a nice divisor of the form $4 \cdot 9$, $4 \cdot 15$ or $4 \cdot 25$. For $n = 36$ we have another solution, in other two cases we have inequality $f(n) < n$. If we want to seek new solution in this case, since they cannot be nice multiples of 60 and 100, the only possibility is that almost perfect number has nice divisor $4 \cdot 27$. But we have (equality case in lemma) that $f(4 \cdot 27) < 3f(4 \cdot 9) = 4 \cdot 27$. So, there are no more solutions in this case.

Fourth case: $k = 3$

For any odd prime we have $f(8p) = 3f(8) = 30$. Similarly to other cases, we only observe candidates of the form $8 \cdot 3^l$. Number $8 \cdot 3$ is not almost perfect, all other candidates have nice divisor $8 \cdot 9$. But, we have $f(72) = 60 < 72$. As we always concluded, we do not have any new solutions.

So, all almost perfect numbers are $3, 18, 36$.

Problem 4. Let ABC be an acute angled triangle. Let B', A' be points on the perpendicular bisectors of AC, BC respectively such that $B'A \perp AB$ and $A'B \perp AB$. Let P be a point on the segment AB and O the circumcenter of the triangle ABC. Let D, E be points on BC, AC respectively such that $DP \perp BO$ and $EP \perp AO$. Let O' be the circumcenter of the triangle CDE. Prove that B', A' and O' are collinear.

(Steve Dinh)

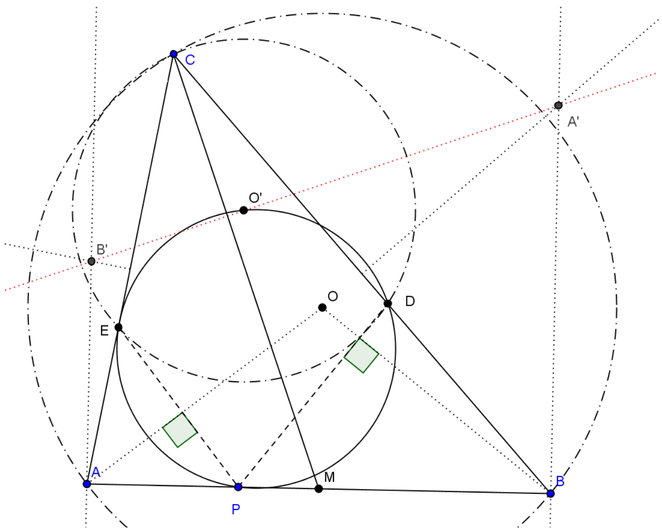
Solution. *Remark* We first start by giving some intuition on how the problem can be approached. We won't go into detail here but do give partial marks for correct ideas. We believe that any essentially correct solution should have them in the background so we don't require them to be written down explicitly.

We notice that if $P \equiv A$ then $O' \equiv B'$ while if $P \equiv B$ we have $O' \equiv A'$. So the problem is equivalent to showing that as P varies on the segment AB respective O' map to a segment and we are now interested in identifying this segment.

It is hence natural to draw a picture not containing anything dependent on P and try to identify the line $A'B'$. Which turns out to be perpendicular to CM where M is the midpoint of AB.

Furthermore we note that $B'M^2 - B'C^2 = AM^2 = A'M^2 - A'C^2$ and this defines the line uniquely (and shows $A'B' \perp CM$).

The following sketch represents the problem setting when we do include the elements depending on P.

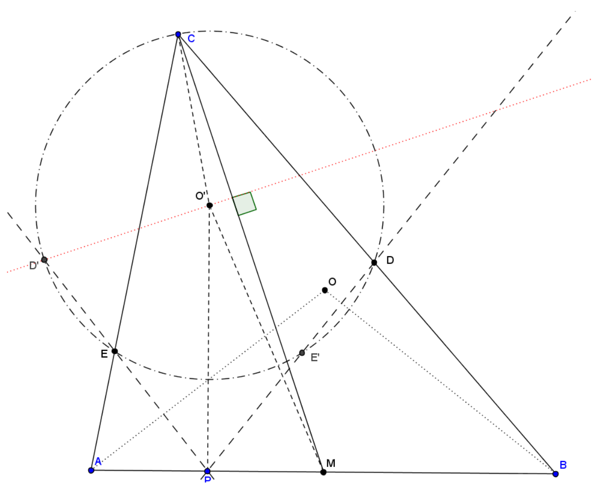


We now start with the formal proof.

It is enough to show that $O'M^2 - O'C^2 = AM^2$ for all P, including $P = A, B$. Which allows us to draw the following sketch omitting B', C'.

We first prove that $O'EPD$ is a cyclic quadrilateral. This follows as $EO'D = 2ACB = APE + BPD = \pi - EPD$ as $ACB = APE = BPD$. This in turn implies PO' is an angle bisector of the angle EPD and $PO' \perp AB$.

We now have all the ingredients to show $O'M^2 - O'C^2 = AM^2$. The following sketch illustrates the last part of the proof.



We introduce the point D' as the second intersection of the line PE and the circumcircle of CDE so that $O'P^2 - O'C^2 = PE \cdot PD'$.

Now as PO' is the angle bisector of EPD we have $PD = PD'$ by the extended $S-S-K$ congruency theorem and the following observation. There is some care needed here, mainly the options we get by $S-S-K$ are $PD = PD'$ or

$PD = PE$ but if $PD = PE$ triangles $P'EO'$ and $P'DO'$ are congruent by $S-S-S$ congruency theorem so in particular $EO'P = DO'P = CAB$ while $EPO' = DPO' = \frac{\pi}{2} - CAB$ so PD and PE are tangents so in fact $D' \equiv E$ so the above claim is still true.

Now noticing triangles APE and BPD are similar we get $\frac{PE}{AP} = \frac{PB}{PD}$ implying $AP \cdot BP = PE \cdot PD = PE \cdot PD'$

As $PO' \perp AB$ by using pythagoras theorem we get $O'M^2 - O'C^2 - AM^2 = O'P^2 - O'C^2 + PM^2 - AM^2 = PD' \cdot PE - AP \cdot BP = 0$. Where we used $O'P^2 - O'C^2 = PE \cdot PD'$ by the power of the point P to the circumcircle of CDE and $AM^2 - PM^2 = (BM + PM)(AM - PM) = AP \cdot PB$.

This completes the proof. ■