2^{ND} European Mathematical Cup

7^{th} December 2013–15^{st} December 2013

Junior Category

Problems and Solutions

Problem 1. For a positive integer m let $m?$ be the product of first m prime numbers.
Determine if there exist positive integers m and n with the following property:

$$m? = n(n+1)(n+2)(n+3).$$

(Matko Ljulj)

Solution. Such numbers don't exist.
Let's assume the contrary i.e. there are such m and n.
We can note that there is only one prime divisible by 2 and that it 2 itself thus $m?$ isn't divisible by 4. On the other hand, the product $n(n+1)(n+2)(n+3)$ is product of 4 consecutive integers so two of them are even making the product divisible by 4.
Thus equality $m? = n(n+1)(n+2)(n+3)$ gives us a contradiction as LHS is not divisible by 4 while RHS is.

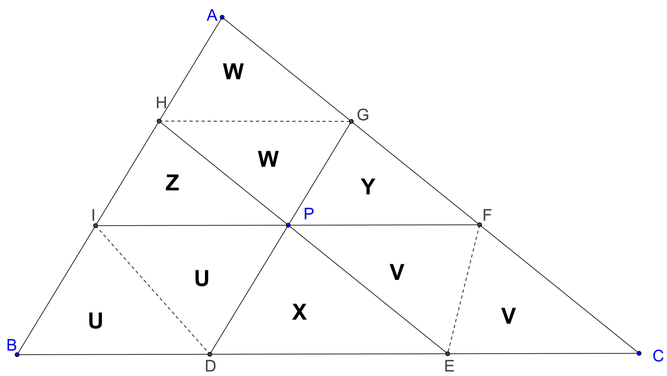
Problem 2. Let P be a point inside a triangle ABC. A line through P parallel to AB meets BC and CA at points L and F, respectively. A line through P parallel to BC meets CA and BA at points M and D respectively, and a line through P parallel to CA meets AB and BC at points N and E respectively. Prove

$$(PDBL) \cdot (PECM) \cdot (PFAN) = 8 \cdot (PFM) \cdot (PEL) \cdot (PDN),$$

where (XYZ) and $(XYZW)$ denote the area of the triangle XYZ and the area of quadrilateral $XYZW$.

(Steve Dinh)

Solution.



Let's denote the areas as on the sketch.
The problem is equivalent to

$$U \cdot V \cdot W = X \cdot Y \cdot Z.$$

Let x and y be lengths of altitudes from I and D in the triangle BID and let a and b be lenghts of sides BI and BD. We can deduce

$$X = (PED) = \frac{1}{2} \cdot a \cdot y \cdot \frac{BC}{BA},$$

$$Z = (PIH) = \frac{1}{2} \cdot b \cdot x \cdot \frac{BA}{BC} \text{ and}$$

$$U = (BID) = \frac{1}{2} \cdot a \cdot x = \frac{1}{2} \cdot b \cdot y$$

This gives $U^2 = X \cdot Z$. Analogously we get $W^2 = Y \cdot Z$ and $V^2 = X \cdot Y$. Multiplying all three equalities we get the desired equation.

Second solution. Let's denote the areas of triangles PEL, PFM, PDN as P_A, P_B, P_C respectively and let's denote the areas of quadrilaterals $PFAN$, $PDBL$, $PECM$ as Q_A, Q_B, Q_C respectively. We want to prove $Q_AQ_BQ_C = 8P_AP_BP_C$. Triangles PEL, PFM, and PDN are similar to the triangle ABC (they have respective pairs of sides on parallel lines). Let's denote the respective similarity coefficients as k_A, k_B, k_C. As triangles PEL, PFM, and PDN are in the interior of ABC, all those coefficients are less than 1.
Triangle ENB is similar to the triangle ABC. Its similarity coefficient is

$$\frac{EN}{AC} = \frac{EF + FN}{AC} = \frac{EF}{AC} + \frac{FN}{AC} = k_A + k_C.$$

From all these similarity relations we get area relations. Namely:

$$P_A : P_B = (P_A : (ABC)) : (P_B : (ABC)) = \left(\frac{k_A}{k_B}\right)^2 \implies P_A = \left(\frac{k_A}{k_B}\right)^2 P_B,$$
$$P_C : P_B = (P_C : (ABC)) : (P_B : (ABC)) = \left(\frac{k_C}{k_B}\right)^2 \implies P_C = \left(\frac{k_C}{k_B}\right)^2 P_B.$$

Using this we get:

$$(P_A + P_C + Q_B) : P_B = (ENB) : (PFM) = (k_A + k_C)^2 : (k_B)^2$$
$$\implies P_A + P_C + Q_B = \frac{k_A^2 + 2k_Ak_C + k_C^2}{k_B^2} P_B = \frac{k_A^2}{k_B^2} P_B + \frac{2k_Ak_C}{k_B^2} P_B + \frac{k_C^2}{k_B^2} P_B = P_A + \frac{2k_Ak_C}{k_B^2} P_B + P_C$$
$$\implies Q_B = \frac{2k_Ak_C}{k_B^2} P_B.$$

Similary by the same process applied to FLC and MDA we get $Q_C = \frac{2k_Bk_A}{k_C^2} P_C$ i $Q_A = \frac{2k_Ck_B}{k_A^2} P_A$. Multiplying what we got we have

$$Q_AQ_BQ_C = \frac{2k_Ck_B}{k_A^2} P_A \frac{2k_Ak_C}{k_B^2} P_B \frac{2k_Bk_A}{k_C^2} P_C = 8 \frac{k_A^2k_B^2k_C^2}{k_A^2k_B^2k_C^2} P_AP_BP_C = 8P_AP_BP_C,$$

Q.E.D.

Problem 3. We are given a combination lock consisting of 6 rotating discs. Each disc consists of digits $0, 1, 2, \ldots, 9$, in that order (after digit 9 comes 0). Lock is opened by exactly one combination. A move consists of turning one of the discs one digit in any direction and the lock opens instantly if the current combination is correct. Discs are initially put in the position 000000, and we know that this combination is not correct.

a) What is the least number of moves necessary to ensure that we have found the correct combination?
b) What is the least number of moves necessary to ensure that we have found the correct combination, if we know that none of the combinations 000000, 111111, 222222, ..., 999999 is correct?

(Ognjen Stipetić, Grgur Valentić)

Solution. We will solve the subproblems seperately.

a) In order to ensure that we have discovered the code we need to check all but one of the combinations (as otherwise all unchecked codes can be the correct combination). Total number of combinations is 10^6 (as each of the 6 discs consists of 10 digits). As we are given that 000000 is not the correct combination we require at least $10^6 - 2$ moves. We will now prove that there is a sequence of $10^6 - 2$ moves each checking a different combination. We will prove this by induction on the number of wheels where the case $n = 6$ is given in the problem.

Claim: For a lock of n wheels and for any starting combination of the wheels $(a_1a_2 \ldots a_n)$ there is a sequence of moves checking all 10^6 combinations exactly once, for all $n \in \mathbb{N}$.

Basis: For $n = 1$ and for the starting combination (a), we consider the sequence of moves

$$a \to a + 1 \to a + 2 \to \ldots \to 9 \to 0 \to 1 \to \ldots \to a - 1$$

Assumption: The induction claim is valid for some $n \in \mathbb{N}$.
Step: We will prove that the claim holds for $n+1$ as well. We consider an arbitrary starting state $(a_1 a_2 \dots a_n a_{n+1})$. By the induction hypothesis there is a sequence of moves such that starting from this state we can check all the states showing a_{n+1} on the last disc. Let this sequence of moves end with the combination $(b_1 b_2 \dots b_n a_{n+1})$.
Now we make the move $(b_1 b_2 \dots b_n a_{n+1}) \to (b_1 b_2 \dots b_n a_{n+1}+1)$ (if a_{n+1} is 9, then we turn the disc to show 0).
We continue in the same way applying the induction hypothesis on first n discs and the rotation the $n+1$-st disc. This way we get the sequence of moves

$$\begin{aligned}(a_1 a_2 \dots a_n a_{n+1}) &\to (b_1 b_2 \dots b_n a_{n+1}) \to (b_1 b_2 \dots b_n a_{n+1}+1)\\ &\to (c_1 c_2 \dots c_n a_{n+1}+1) \to (c_1 c_2 \dots c_n a_{n+1}+2)\\ &\dots\\ &\to (j_1 j_2 \dots j_n a_{n+1}-2) \to (j_1 j_2 \dots j_n a_{n+1}-1).\end{aligned}$$

This sequence checks each combination exactly once finishing the induction and proving our claim.

b) As in the a) part, we conclude that we have to check all the combinations apart from 000000, 111111, ..., 999999 and we can be sure as to what is the solution before the move checking the last combination.
We denote the combination as *black* if the sum of its digits is even and *white* if that sum is odd. We can notice that all the combinations 000000, 111111, ..., 999999 are black and by each move we swap the color of the current combination.
Number of black combinations all of which we need to check at least once is $\frac{10^6}{2} - 10$ while number of such white combinations is $\frac{10^6}{2}$.
As we are checking white combinations every second move, in order to check all $\frac{10^6}{2}$ white combination swe need at least $2\frac{10^6}{2} - 1 = 10^6 - 1$ moves, thus we need at least $10^6 - 2$ moves to find the correct combination.
An example doing this in $10^6 - 2$ moves has been given in part a).

Problem 4. Let a, b, c be positive real numbers satisfying

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geqslant \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a}.$$

Prove

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + a + b + c + 2 \geqslant 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}).$$

(Dimitar Trenevski)

Solution. We start with the given condition:

$$\begin{gathered}\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geqslant \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a} \iff\\ \frac{a+ab+bc}{1+b+c} + \frac{b+bc+ba}{1+c+a} + \frac{c+ca+cb}{1+a+b} \geqslant \frac{ab+ac+bc}{1+a+b} + \frac{bc+ab+bc}{1+b+c} + \frac{ca+bc+ab}{1+c+a} \iff\\ \frac{a(1+b+c)}{1+b+c} + \frac{b(1+c+a)}{1+c+a} + \frac{c(1+a+b)}{1+a+b} \geqslant \frac{ab+bc+ca}{1+a+b} + \frac{ab+bc+ca}{1+b+c} + \frac{ab+bc+ca}{1+c+a} \iff\\ a+b+c \geqslant (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right).\end{gathered}$$

Now using *Cauchy-Schwarz* inequality we get:

$$\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(c(1+a+b) + a(1+b+c) + b(1+c+a)) \geqslant (\sqrt{a} + \sqrt{b} + \sqrt{c})^2.$$

Combining the last two inequalities we get:

$$\begin{gathered}(a+b+c)(a+b+c+2(ab+bc+ca)) \geqslant\\ \geqslant (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(a+b+c+2(ab+bc+ca)) =\\ = (ab+bc+ca)\left(\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a}\right)(c(1+a+b) + a(1+b+c) + b(1+c+a)) \geqslant\\ \geqslant (ab+bc+ca)(\sqrt{a} + \sqrt{b} + \sqrt{c})^2,\end{gathered}$$

which now by some algebraic manipulation gives:

$$
\begin{gathered}
(a+b+c)(a+b+c+2(ab+bc+ca)) \geqslant (ab+bc+ca)(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \iff \\
(a+b+c)^2+2(a+b+c)(ab+bc+ca) \geqslant (ab+bc+ca)(a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})) \iff \\
(a^2+b^2+c^2)+(2(a+b+c)+2)(ab+bc+ca) \geqslant (ab+bc+ca)(a+b+c+2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})) \iff \\
\frac{a^2+b^2+c^2}{ab+bc+ca}+a+b+c+2 \geqslant 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}),
\end{gathered}
$$

where the last inequality is exactly the one we wanted to prove.

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.