1^{ST} European Mathematical Cup

24^{th} November 2012–1^{st} December 2012

Junior Category

Problem 1. Let ABC be a triangle and Q a point on the internal angle bisector of $\angle BAC$. Circle ω_1 is circumscribed to triangle BAQ and intersects the segment AC in point $P \neq C$. Circle ω_2 is circumscribed to the triangle CQP. Radius of the cirlce ω_1 is larger than the radius of ω_2. Circle centered at Q with radius QA intersects the circle ω_1 in points A and A_1. Circle centered at Q with radius QC intersects ω_1 in points C_1 and C_2. Prove $\angle A_1BC_1 = \angle C_2PA$.

(Matija Bucić)

Problem 2. Let S be the set of positive integers. For any a and b in the set we have $GCD(a,b) > 1$. For any a, b and c in the set we have $GCD(a,b,c) = 1$. Is it possible that S has 2012 elements?

$GCD(x,y)$ and $GCD(x,y,z)$ stand for the greatest common divisor of the numbers x and y and numbers x, y and z respectively.

(Ognjen Stipetić)

Problem 3. Do there exist positive real numbers x, y and z such that

$$x^4 + y^4 + z^4 = 13,$$
$$x^3y^3z + y^3z^3x + z^3x^3y = 6\sqrt{3},$$
$$x^3yz + y^3zx + z^3xy = 5\sqrt{3}?$$

(Matko Ljulj)

Problem 4. Let k be a positive integer. At the European Chess Cup every pair of players played a game in which somebody won (there were no draws). For any k players there was a player against whom they all lost, and the number of players was the least possible for such k. Is it possible that at the Closing Ceremony all the participants were seated at the round table in such a way that every participant was seated next to both a person he won against and a person he lost against.

(Matija Bucić)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.