1^{ST} EUROPEAN MATHEMATICAL CUP
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Junior Category

Problems and solutions

Problem 1. Let ABC be a triangle and Q a point on the internal angle bisector of $\angle BAC$. Circle ω_1 is circumscribed to triangle BAQ and intersects the segment AC in point $P \neq C$. Circle ω_2 is circumscribed to the triangle CQP. Radius of the cirlce ω_1 is larger than the radius of ω_2. Circle centered at Q with radius QA intersects the circle ω_1 in points A and A_1. Circle centered at Q with radius QC intersects ω_1 in points C_1 and C_2. Prove $\angle A_1BC_1 = \angle C_2PA$.

(Matija Bucić)

Solution. From the conditions in the problem we have $|QC_1| = |QC_2|$ and $|QA| = |QA_1|$. Also as Q lies on the internal angle bisector of$\angle CAB$ we have $\angle PAQ = \angle QAB \implies |QP| = |QB|$.

Now noting from this that pairs of points A and A_1, C_1 and C_2, B and P are symmetric in line QS_1, where S_1 is the center of ω_1. We can directly conclude $\angle A_1BC_1 = \angle APC_2$ as these is the image of the angle in symmetry.

This way we have avoided checking many cases but there are many ways to prove this problem.

Problem 2. Let S be the set of positive integers. For any a and b in the set we have $GCD(a, b) > 1$. For any a, b and c in the set we have $GCD(a, b, c) = 1$. Is it possible that S has 2012 elements?

$GCD(x, y)$ and $GCD(x, y, z)$ stand for the greatest common divisor of the numbers x and y and numbers x, y and z respectively.

(Ognjen Stipetić)

Solution. There is such a set.

We will construct it in the following way: Let $a_1, a_2, \dots a_{2012}$ equal to 1 in the begining. Then we take $\frac{2012\cdot 2011}{2}$ different prime numbers, and assign a different prime to every pair a_i, a_j (where $i \neq j$) and multiply them with this assigned number. (I.e. for the set of 4 elements we can take 2, 3, 5, 7, 11, 13, so S would be $\{2\cdot 3\cdot 5, 2\cdot 7\cdot 11, 3\cdot 7\cdot 13, 5\cdot 11\cdot 13\}$.

The construction works as we have multiplied any pair of numbers with some prime so the condition $gcd(a, b) > 1$ is satisfied for all a, b. As well as each prime divides exactly 2 primes so no three numbers a, b, c can have $gcd(a, b, c) > 1$.

Problem 3. Do there exist positive real numbers x, y and z such that

$$x^4 + y^4 + z^4 = 13,$$
$$x^3y^3z + y^3z^3x + z^3x^3y = 6\sqrt{3},$$
$$x^3yz + y^3zx + z^3xy = 5\sqrt{3}?$$

(Matko Ljulj)

Solution. Let's assume that such x, y, z exist. Let $a = x^2$, $b = y^2$, $c = z^2$. As well, let $A = a + b + c$, $B = ab + bc + ca$, $C = abc$. The upper system can be rewritten as:

$$a^2 + b^2 + c^2 = 13 \implies (a + b + c)^2 - 2(ab + bc + ca) = 13 \implies A^2 - 2B = 13$$
$$xyz(x^2y^2 + y^2z^2 + z^2x^2) = 6\sqrt{3} \implies \sqrt{C}B = 6\sqrt{3}$$
$$xyz(x^2 + y^2 + z^2) = 5\sqrt{3} \implies \sqrt{C}A = 5\sqrt{3}.$$

We can note that a, b and c are positive reals (They are not negaitve from the definition; and as $\sqrt{C}B = 6\sqrt{3}$ they are not 0).

When we cancel out $\sqrt{C}$ from the second and third equation we get $5B = 6A$. When we express B in terms of A and put int the first equation we get a quadratic equation

$$A^2 - \frac{12}{5}A - 13 = 0.$$

with solutions 5 and $-\frac{13}{5}$. As a, b and c are positive reals, and the sum must be positive so their sum is poistive real number as well. So $A = 5 \implies B = 6 \implies C = 3$.

By *AM-GM* inequality we get

$$\begin{aligned} \frac{ab+bc+ca}{3} &\geqslant \sqrt[3]{ab \cdot bc \cdot ca} \\ \iff \frac{B}{3} &\geqslant \sqrt[3]{C^2} \\ \iff \frac{6}{3} &\geqslant \sqrt[3]{9} \;/^3 \\ \iff 8 &\geqslant 9. \end{aligned}$$

so we reached a contradiction, thus such x, y, z don't exist.

Problem 4. Let k be a positive integer. At the European Chess Cup every pair of players played a game in which somebody won (there were no draws). For any k players there was a player against whom they all lost, and the number of players was the least possible for such k. Is it possible that at the Closing Ceremony all the participants were seated at the round table in such a way that every participant was seated next to both a person he won against and a person he lost against.

(Matija Bucić)

Solution. The answer is yes.

In this problem we could use graph theory terminology but as this problem was intended for younger students we shall avoid mentioning any specific graph theory terms.

Let's take the largest number of participants whom we can seat around the table as desired. If we have seated all the participants we are done. Otherwise there is a person not seated at the table. As well there is at least one person seated at the table so let's name it a.

WLOG we can assume that for each person seated at the table to his right there is a person he won against and to his left a person he lost against.

Denote by W the set of people who won against person a, and are not seated at the table. Similarly, let L denote the set of all people who lost against a and are not seated at the table.

Let's consider any person p from W. If person p lost against the left neighbour of a, then we could seat p in between a and his (former) left neighbour, which is a contradiction with the assumption that we have seated the maximal possible number of people. So p won against the left neighbour of a. Using similar deduction we conclude that p won against the next left neighbour as well etc. So p must have won against everybody seated at the table.

In the same way if we consider any person q from L and consider the right neighbour of a, we can conclude that q lost against every person seated at the table.

If some person r from W lost against some person s in L, then instead of seating a we can seat s and r respectively by which we would reach a contradiction to the number of people seated being maximal.

So we conclude that all the people in W won against all people not in W and all the people in L lost against all people not in L.

As there is a someone who is not seated either W or L is non-empty. If W is non-empty, we can consider the set W as an independent chess cup. It is a cup with smaller number of participants but still satisfying problem conditions which would be the contradiction with the fact that our starting cup is the smallest such cup.

As well if L is non-empty, the smaller cup made by people seated at the table and people in W also satisfies the problem conditions and gives us a contradiction.

So the only possibility is that both W and L are empty so indeed it is possible to seat everyone at such table.