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1^{st} BALKAN STUDENT MATHEMATICAL COMPETITION

1. Matematičko natjecanje učenika Balkana

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Solutions

Problem 1. If x, y and z are positive real numbers for which $x+y+z=1$, prove the inequality

$$\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{y+z}}+\frac{1}{\sqrt{z+x}} \leq \frac{1}{\sqrt{2xyz}}.$$

(Adrian Satja Kurdija)

Solution. Multiplying the given inequality by $2\sqrt{xyz}$, we get an equivalent inequality

$$2\sqrt{\frac{xyz}{x+y}}+2\sqrt{\frac{xyz}{y+z}}+2\sqrt{\frac{xyz}{z+x}} \leq \sqrt{2}.$$

Let us notice that $\dfrac{xy}{x+y} \leq \dfrac{x+y}{4} \iff (x-y)^2 \geq 0$, so $\dfrac{xy}{x+y} \leq \dfrac{x+y}{4}$. Now,

$$2\sqrt{\frac{xyz}{x+y}} = 2\sqrt{\frac{xy}{x+y}\cdot z} \leq 2\sqrt{\frac{x+y}{4}\cdot z} = \sqrt{z\left(x+y\right)}.$$

(2 points)

Using Arithmetic Mean - Geometric Mean inequality (or using the fact that the square of a real number is always nonnegative) on numbers $z\sqrt{2}$ and $\dfrac{x+y}{\sqrt{2}}$, we get

$$\sqrt{z\left(x+y\right)} = \sqrt{z\sqrt{2}\cdot\frac{x+y}{\sqrt{2}}} \leq \frac{1}{2}\left(z\sqrt{2}+\frac{x+y}{\sqrt{2}}\right) = \frac{\left(x+y+2z\right)\sqrt{2}}{4}.$$

(3 points)

With this, we have shown that

$$2\sqrt{\frac{xyz}{x+y}} \leq \frac{\left(x+y+2z\right)\sqrt{2}}{4}.$$

Analogously, we show that

$$\begin{aligned} 2\sqrt{\frac{xyz}{y+z}} &\leq \frac{\left(y+z+2x\right)\sqrt{2}}{4}, \\ 2\sqrt{\frac{xyz}{z+x}} &\leq \frac{\left(z+x+2y\right)\sqrt{2}}{4}. \end{aligned}$$

By adding these three inequalities we get

$$2\sqrt{\frac{xyz}{x+y}}+2\sqrt{\frac{xyz}{y+z}}+2\sqrt{\frac{xyz}{z+x}} \leq \frac{\left(4x+4y+4z\right)\sqrt{2}}{4} = \sqrt{2}.$$

We have, hence, proven the inequality in question. (5 points)

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Problem 2. A natural number is written in each cell of 10×10 table. It is known that, no matter which 5 columns and 5 rows of this table we choose, the sum of numbers in their 25 intersection cells is even. Prove that all the numbers in the table are even.

Solution. Let us first prove the following lemma.

Lemma 1. If the sum of each 5 of the given 10 natural numbers is even, then all these numbers are even.

Proof. Let's assume the opposite. It is clear that not all numbers can be odd. Therefore, there has to be at least one odd and at least one even number. Then, if there are 5 or more odd numbers, by choosing 5 odd numbers we reach a contradiction. If there are less than 5 odd numbers, by choosing 4 even and one odd number, we also reach a contradiction. Hence, this lemma is proven. (2 points)

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Let us observe any 5 columns of the given table. For every row of the table, let's compute the sum of numbers in cells which we get by intersecting the row with these 5 columns. That way we get 10 natural numbers (one for each row). The sum of any 5 of these numbers is even (this follows from the conditions of the problem). Now, using **Lemma 1**, we conclude that each of these 10 sums is even. (4 points)

By observing all possible choices of 5 columns of the given table, we get that the sum of each 5 numbers of every row is even. Again, using **Lemma 1**, we conclude that every number in each row is even. Therefore, all numbers in the table is even. (4 points)

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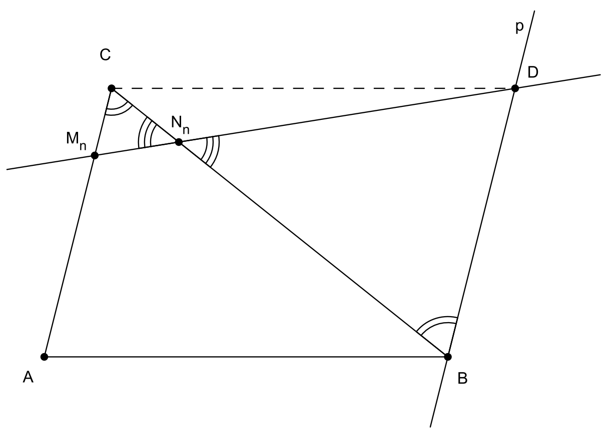
Problem 3. Let M_n and N_n be points on sides $\overline{CA}$ and $\overline{CB}$ of triangle ABC, respectively, such that

$$|CM_n| = \frac{1}{n}|CA|, \quad |CN_n| = \frac{1}{n+1}|CB|, \quad \forall n \in \mathbb{N}.$$

Find the locus of points $M_iN_i \cap M_jN_j$, where i and j are different natural number.

Note. $M_iN_i \cap M_jN_j$ denotes the intersection of lines M_iN_i and M_jN_j.

Solution. We intend to show that all the mentioned lines intersect in one point. That will be point D such that quadrilateral $ABDC$ is a parallelogram. (1 point)



Let n be any natural number. We draw line p parallel to line AC such that B is on p . Let D be the intersection of p i M_nN_n. Let us notice that $\measuredangle M_nCN_n = \measuredangle ACB = \measuredangle CBD = \measuredangle N_nBD$ (alternate interior angles) and $\measuredangle CN_nM_n = \measuredangle BN_nD$ (vertical angles). So, we have shown that triangles CM_nN_n and BDN_n are similar (they have two equal angles). (2 points)

Now,

$$\frac{|BD|}{|CM_n|} = \frac{|BN_n|}{|CN_n|},$$

which means

$$\begin{aligned}
|BD| &= \frac{|CM_n| \cdot |BN_n|}{|CN_n|} = \frac{\frac{1}{n} \cdot |CA| \cdot (|CB| - |CN_n|)}{|CN_n|} \\
&= \frac{\frac{1}{n}|CA| \cdot \left(|CB| - \frac{1}{n+1}|CB|\right)}{\frac{1}{n+1}|CB|} \\
&= \frac{\frac{1}{n}|CA| \cdot \frac{n}{n+1}|CB|}{\frac{1}{n+1}|CB|} \\
&= |CA|.
\end{aligned}$$

With this, we have shown that $|BD| = |AC|$ and, since we know that $BD \parallel AC$, it follows that quadrilateral $ABDC$ is a parallelogram. (4 points)

Now we know that line M_nN_n goes through point D such that quadrilateral $ABDC$ is a parallelogram for each natural number n. Finally, we conclude that the locus of $M_iN_i \cap M_jN_j$, where i and j are different natural numbers, is point D such that quadrilateral $ABDC$ is a parallelogram. (3 points)

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Problem 4. a, b and c are natural numbers. It is known that $a^2 + b^2 + abc$ has no more than 2008 natural divisors and that it is divisible by $(c+2)^{1004}$. Prove that a and b are not relatively prime.

(Adrian Satja Kurdija)

Solution. Let $A = a^2 + b^2 + abc$ and $p = c + 2$, having in mind that then $p \geq 3$. Let's assume that there exists a prime number q such that $q^2 \mid p$. Then, $q^{2008} \mid A$, and as q^{2008} itself has $2009 > 2008$ divisors, we reach a contradiction. So, there does not exist a prime number q such that $q^2 \mid p$. Further, let us assume that there exist different prime numbers r and s which divide p. Then, $r^{1004}s^{1004} \mid A$ and, obviously, $1005^2 > 2008$ and we, again, have a contradiction. Hence, there do not exist two different prime numbers which both divide p. With all of this, we have shown that p is prime. (2 points)

Now, $c = p - 2$, and, for that reason,

$$A = a^2 + b^2 + abc = a^2 + b^2 + ab(p-2) = a^2 - 2ab + b^2 + abp = (a-b)^2 + abp.$$

Since $p \mid A$ and $p \mid abp$, it follows that $p \mid (a-b)^2$ and, since p is prime, it further follows that $p \mid a - b$. From this, we conclude finally that $p^2 \mid (a-b)^2$. (2 points)

Furthermore, $p^2 \mid A$ and $p^2 \mid (a-b)^2$, so $p^2 \mid abp$, which leads to $p \mid ab$, must also be true. (1 point)

Now we know that $p \mid a - b$, $p \mid ab$ and that p is prime. Since $p \mid ab$, this means that p divides at least one of the numbers a and b. (1 point)

Without loss of generality, we may assume that $p \mid a$. Then, from $p \mid a - b$, it directly follows that $p \mid b$. So, numbers a and b are not relatively prime. (4 points)

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