3RD EUROPEAN MATHEMATICAL CUP

6th December 2014–14th December 2014

Senior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problem 1. Prove that there are infinitely many positive integers which can't be expressed as $a^{d(a)} + b^{d(b)}$ where a and b are positive integers.

For positive integer a expression $d(a)$ denotes the number of positive divisors of a. *(Borna Vukorepa)*

Problem 2. Jeck and Lisa are playing a game on an $m \times n$ board, with $m, n > 2$. Lisa starts by putting a knight onto the board. Then in turn Jeck and Lisa put a new piece onto the board according to the following rules:

1. Jeck puts a queen on an empty square that is two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from Lisa's last knight.
2. Lisa puts a knight on an empty square that is on the same, row, column or diagonal as Jeck's last queen.

The one who is unable to put a piece on the board loses the game. For which pairs (m, n) does Lisa have a winning strategy?

(Stijn Cambie)

Problem 3. Let $ABCD$ be a cyclic quadrilateral with the intersection of internal angle bisectors of $\angle ABC$ and $\angle ADC$ lying on the diagonal AC. Let M be the midpoint of AC. The line parallel to BC that passes through D intersects the line BM in E and the circumcircle of $ABCD$ at F where $F \neq D$. Prove that $BCEF$ is a parallelogram.

(Steve Dinh)

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following holds:

$$f(x^2) + f(2y^2) = (f(x + y) + f(y))(f(x - y) + f(y)).$$

(Matija Bucić)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.