



11TH EUROPEAN MATHEMATICAL CUP

10th December 2022 - 18th December 2022

Senior Category



Problem 1. Let $n \geq 3$ be a positive integer. Alice and Bob are playing a game in which they take turns colouring the vertices of a regular n -gon. Alice plays the first move. Initially, no vertex is coloured. Both players start the game with 0 points.

On their turn, the player colours a vertex V which hasn't been coloured and gains k points, where k is the number of neighbouring vertices of V which have already been coloured. (Thus, k is either 0, 1 or 2.)

The game ends when all vertices have been coloured, and the player with more points wins. If the players have the same number of points, no one wins. Determine all $n \geq 3$ for which Alice has a winning strategy, and all $n \geq 3$ for which Bob has a winning strategy.

(Josip Pupić)

Problem 2. We say a positive integer n is *lovely* if there exist a positive integer k and (not necessarily distinct) positive integers d_1, d_2, \dots, d_k such that $n = d_1 d_2 \dots d_k$ and

$$d_i^2 \mid n + d_i$$

for all $i \in \{1, \dots, k\}$.

- Are there infinitely many lovely numbers?
- Does there exist a lovely number greater than 1 which is a square of an integer?

(Ivan Novak)

Problem 3. Let \mathbb{R} denote the set of all real numbers. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^3) + f(y)^3 + f(z)^3 = 3xyz$$

for all $x, y, z \in \mathbb{R}$ such that $x + y + z = 0$.

(Kyprianos-Iason Prodromidis)

Problem 4. Five points A, B, C, D and E lie on a circle τ clockwise in that order such that AB is parallel to CE and $\angle ABC > 90^\circ$. Let k be a circle tangent to AD, CE and τ such that the circles k and τ touch on the arc \widehat{ED} which doesn't contain A, B and C . Let $F \neq A$ be the intersection of τ and the line tangent to k passing through A different from AD .

Prove that there exists a circle tangent to BD, BF, CE and τ .

(Steve Vo Dinh)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.