11^{TH} European Mathematical Cup

10^{th} December 2022 - 18^{th} December 2022

Junior Category

Problem 1. Find all positive integers n for which there exist three positive divisors a, b, c of n such that $a > b > c$ and

$$a^2 - b^2,\ \ b^2 - c^2,\ \ a^2 - c^2$$

are also divisors of n.

(Kims Georgs Pavlovs)

Problem 2. Find all pairs of positive real numbers (x, y) such that xy is an integer and

$$x + y = \lfloor x^2 - y^2 \rfloor.$$

Remark: For a real number z, we denote by $\lfloor z \rfloor$ the greatest integer not greater than z. Some examples are $\lfloor 17 \rfloor = 17$, $\lfloor 8.2 \rfloor = 8$ and $\lfloor -5.4 \rfloor = -6$.

(Ivan Novak)

Problem 3. Let ABC be an acute-angled triangle with $|BC| < |AC|$. Let I be the incenter and τ the incircle of ABC, which touches BC and AC at points D and E, respectively. The point M is on τ such that BM is parallel to DE and M and B are on the same side of the angle bisector of $\angle BCA$. Let F and H be the intersections of τ with BM and CM different from M, respectively. Let J be a point on the line AC such that JM is parallel to EH. Let K be the intersection of JF and τ different from F.

Prove that the lines ME and KH are parallel.

(Steve Vo Dinh)

Problem 4. Let $X = \{1, 2, 3, \ldots, 300\}$. A collection F of distinct (not necessarily non-empty) subsets of X is *lovely* if for any three (not necessarily distinct) sets A, B, C in F, at most three out of the following eight sets are non-empty:

$$\begin{array}{cccc} A \cap B \cap C, & \overline{A} \cap B \cap C, & A \cap \overline{B} \cap C, & A \cap B \cap \overline{C}, \\ \overline{A} \cap \overline{B} \cap C, & \overline{A} \cap B \cap \overline{C}, & A \cap \overline{B} \cap \overline{C}, & \overline{A} \cap \overline{B} \cap \overline{C}, \end{array}$$

where $\overline{S}$ denotes the set of all elements of X that are not in S.

What is the greatest possible number of sets in a lovely collection?

(Miroslav Marinov)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.