10TH EUROPEAN MATHEMATICAL CUP

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Senior Category

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Problems and Solutions

Problem 1. Alice drew a regular 2021-gon in the plane. Bob then labelled each vertex of the 2021-gon with a real number, in such a way that the labels of consecutive vertices differ by at most 1. Then, for every pair of non-consecutive vertices whose labels differ by at most 1, Alice drew a diagonal connecting them. Let d be the number of diagonals Alice drew. Find the least possible value that d can obtain.

(Ivan Novak)

First Solution. Consider the following labelling of the vertices, where the i-th number of the 2021-tuple below is the label of the i-th vertex:

$$(0.5,\ 1.5,\ 2.5, \ldots, 1009.5,\ 1010.5,\ 1010,\ 1009,\ 1008, \ldots, 2,\ 1).$$

It's easy to see that in this case, Alice will draw 2018 diagonals, those connecting the vertices whose pairs of labels are $\{1.5, 1\}, \{2.5, 2\}, \ldots, \{1009.5, 1009\}$ and $\{1.5, 2\}, \{2.5, 3\}, \ldots, \{1009.5, 1010\}$.

3 points.

We now prove that 2018 is the minimum amount of diagonals Alice could have drawn. Call any labelling of a convex n-gon which satisfies the condition that consecutive vertices have labels which differ by at most 1 a *sweet* labelling, and also call the corresponding n-gon sweet.

We will prove by mathematical induction that for every $n \geqslant 3$, in any sweet labelling of an n-gon, there are at least $n-3$ pairs of nonconsecutive vertices whose labels differ by at most 1. The claim is obvious for $n = 3$. Suppose that the claim is true for some positive integer n.

Consider a sweet labelling of some $n+1$-gon P. Consider a vertex v with the maximum label, L. Then both of its neighbouring vertices have labels in the set $[L-1, L]$, which means that their labels differ by at most 1.

1 point.

Then the n-gon P' obtained from P by erasing v and connecting its neighbouring vertices is also sweet.

2 points.

Applying the inductive hypothesis on it, there are at least $n-3$ pairs of nonconsecutive vertices of P' whose labels differ by at most 1. Adding the pair of neighbours of v, we conclude that P has at most $n-2$ pairs of such vertices. This completes the step of the induction.

4 points.

Second Solution. The example which achieves the desired bound is the same as in the previous solution.

3 points.

Let the labels of the vertices of the 2021-gon be $v_1, \ldots, v_{2021}$, where we assume the labels to be ordered so that $v_1 \leqslant \ldots \leqslant v_{2021}$. Note that this is not necessarily the order in which the values appear on the 2021-gon.

We claim that $|v_i - v_{i+2}| \leqslant 1$, for all $i = 1, 2, \ldots, 2019$.

Assume for the sake of contradiction that there exists an $i \in \{1, 2, \ldots, 2019\}$ for which $v_{i+2} - v_i > 1$. Start a circular walk around the 2021-gon, going from the vertex which has the value v_1, visiting all of the vertices one by one, and returning back to the starting vertex. Doing so visits the values $v_1, \ldots, v_{2021}$ in a certain permuted order, starting and ending on v_1.

We look at the first time during the walk when we step on a value whose index is greater than or equal to $i+2$. Let this index be $j \geqslant i+2$. Let's say that on the previous step, we were on value v_b, where $b \leqslant i+1$. Note that if $b \leqslant i$, then $v_j - v_b \geqslant v_{i+2} - v_i > 1$, so it must be the case that $b = i+1$. Next, we look at the first time we return to an index which is smaller than or equal to i. Such an index must exist since we eventually return back to v_1, and we'll denote it by k. A similar argument as for v_j shows that in the step before reaching v_k, we must have been on index b. This is a contradiction as no vertex can be visited more than once, except for the one we started with.

5 points.

We now have $|v_i - v_j| \leqslant 1$, for all $i = 3, \ldots, 2019$ and $j \in \{i-1, i-2, i+1, i+2\}$, $|v_1 - v_i| \leqslant 1$ for $j \in \{2, 3\}$ and $|v_j - v_{2021}| \leqslant 1$ for $j \in \{2019, 2020\}$, which gives at least $\frac{1}{2}(2+3+4\cdot 2017+3+2) - 2021 = 2018$ diagonals.

2 points.

Third Solution. The example which achieves the desired bound is the same as in the previous solution.

3 points.

Note that

We label the vertices v_1, v_2, …, v_{2021}, and, respectively, their labels x_1, x_2, …, x_{2021} and view the indices modulo 2021. We'll say vertices (v_i, v_j) are a *nigh* pair if their labels differ by at most 1.

Without loss of generality, let 1 and g be the indices among $\{1, 2, \ldots, 2021\}$ of the vertices with the smallest and greatest label, respectively. Without loss of generality we can also assume $g \leqslant 1011$ since we can otherwise mirror the 2021-gon. Additionally, we will assume that $g \geqslant 4$, and the cases $g = 3$ and $g = 2$ are dealt with separately. We make use of the following lemmas.

Lemma 1. *For every $1 < k < g$, there are at least two indices $g \leqslant i, j \leqslant 2022$ such that (v_k, v_i) and (v_k, v_j) are pairs of nigh vertices. Similarly, for every $g < k < 2022$, there are at least two indices $1 \leqslant i, j \leqslant g$ such that (v_k, v_i) and (v_k, v_j) are pairs of nigh vertices.*

Proof. As the statement is obviously symmetric, we will only prove the first half. Let m and M be the smallest and the biggest label, respectively. If $x_k \leqslant m+1$, v_{2021} and v_1 satisfy the condition of the lemma. Similarly, if $x_k \geqslant M-1$, v_g and v_{g+1} satisfy the condition of the lemma. Otherwise, there are at least two indices within the desired range with labels in $[x-1, x+1]$ due to the Intermediate Value Theorem. Namely, if we imagine jumping along the vertices from v_1 do v_g, at the vertex v_1 the label is less than $x-1$ and at the vertex v_g the label is greater than $x+1$. Then at some point in between we must have been at a vertex whose label is from $[x-1, x)$ and at a vertex whose label is from $[x, x+1)$. □

4 points.

Lemma 2. At most one of the vertices v_{2021} and v_2 and at most one of the vertices v_{g-1} and v_{g+1} can be elements of three nigh pairs.

Proof. Clearly, due to Lemma 1, both v_{2021} and v_2 are elements of at least three nigh pairs. Assume both vertices are elements of exactly three nigh pairs. Assume that $x_{2021} \leqslant x_2$. It follows that $x_2 - 1 \leqslant x_1 \leqslant x_{2020} \leqslant x_{2021} + 1 \leqslant x_2 + 1$, therefore, v_{2020}, v_{2021}, v_1 and v_3 are four vertices forming nigh pairs with v_2, a contradiction. We treat the other cases analogously. □

2 points.

Finally, due to Lemma 1, each vertex not neighbouring with g or 1 forms at least 4 nigh pairs. The vertices v_1 and v_g form at least 2 nigh pairs and at most two of their neighbouring vertices form 3 nigh pairs. The minimal number of nigh pairs is therefore $\frac{1}{2} \cdot (2017 \cdot 4 + 2 \cdot 3 + 2 \cdot 2) = 4039$.

1 point.

Assume $g \leqslant 3$. If $g = 2$, all the pairs of vertices are nigh as their labels are all in the set $[x_1, x_2]$. If $g = 3$, each vertex forms a nigh pair with at least one of v_1 or v_3. Without loss of generality, more than half of the remaining vertices form nigh pairs with v_1. But then all of those vertices are nigh as well, so there are at least $\binom{1015}{2} > 4039$ nigh pairs, making this case suboptimal as well.

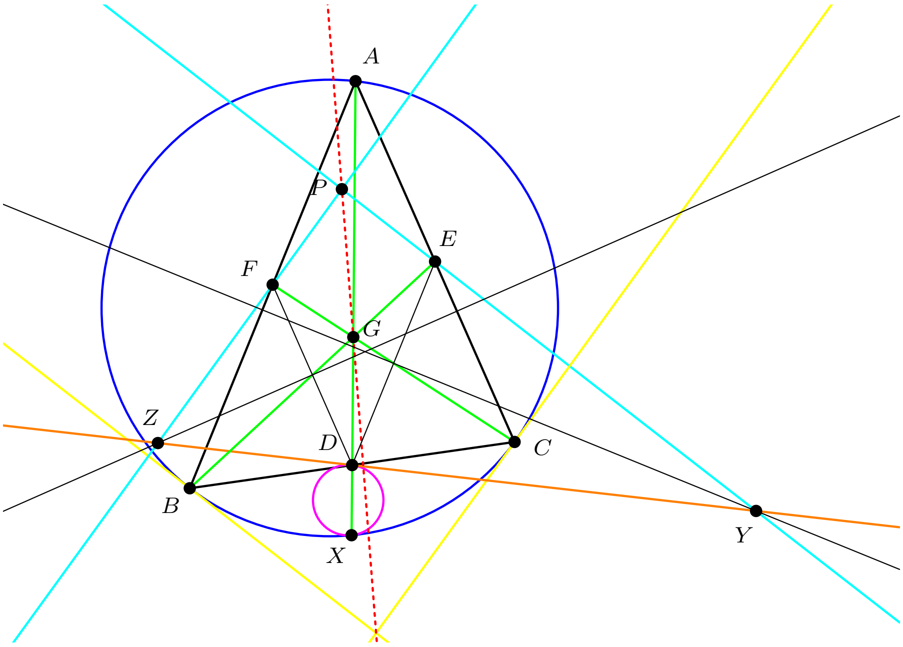
0 points.

Notes on marking:

- Points from different solutions are not additive. Student's score should be the maximum of points scored over all solutions.
- Miscounting the number d in the optimal example or making a similar minor mistake in that part should be awarded **2 points** out of possible **3** for that part of the solution.
- In the first solution, just stating the idea of induction is worth no points on its own.
- In the third solution, dealing with the case $g \leqslant 3$ is worth no points on its own. However, a contestant who doesn't comment these cases can get at most **9 points** for their solution.

Problem 2. Let ABC be a triangle and let D, E and F be the midpoints of sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively. Let $X \neq A$ be the intersection of AD with the circumcircle of ABC. Let Ω be the circle through D and X, tangent to the circumcircle of ABC. Let Y and Z be the intersections of the tangent to Ω at D with the perpendicular bisectors of segments $\overline{DE}$ and $\overline{DF}$, respectively. Let P be the intersection of YE and ZF and let G be the centroid of ABC. Show that the tangents at B and C to the circumcircle of ABC and the line PG are concurrent.

(Jakob Jurij Snoj)



First Solution. Due to the collinearity of A, X and D, there is a homothety at X sending the circumcircle of ABC to Ω. This homothety also sends the tangent at A to the circumcircle of ABC to the tangent at D of Ω.

1 point.

The homothety in G with ratio $-\frac{1}{2}$ sends the circumcircle of ABC to its nine-point circle and the tangent at A to the circumcircle to the tangent at D to the nine-point circle. These tangents are therefore parallel. It follows that Ω and the nine-point circle of ABC share a tangent at D.

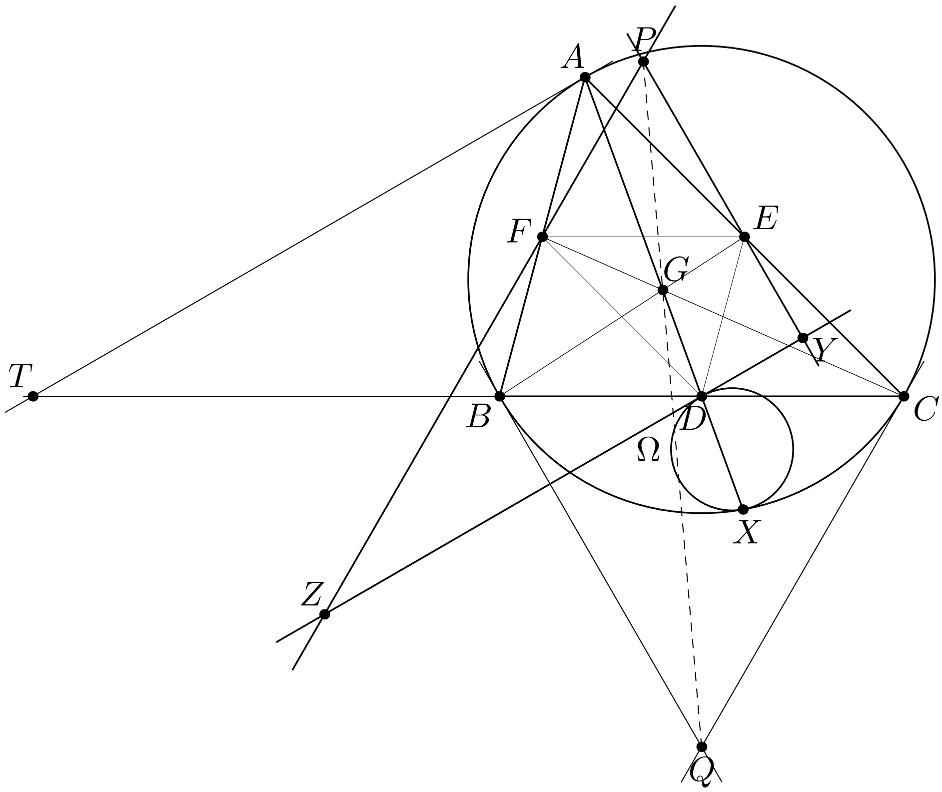
2 points.

As Y and Z lie on the perpendicular bisectors of DE and DF, respectively, it follows that $|YD| = |YE|$ and YE is also tangent to the nine-point circle of ABC - similarly, ZF is also tangent to this circle. We conclude that the nine-point circle of ABC is the incircle of PYZ.

3 points.

Finally, the homothety at G sending the circumcircle of ABC to its nine-point circle sends the tangents through A, B and C to the circumcircle of ABC respectively to the tangents at D, E and F to the nine-point circle of ABC. It, therefore, sends the intersection of tangents at B and C to the circumcircle of ABC to the point P, thus proving the desired collinearity.

4 points.



Second Solution. As in the previous solution, we prove that the tangent at D to Ω is parallel to the tangent at A to the circumcircle of ABC.

1 point.

Let $\sphericalangle BAC = \alpha$, $\sphericalangle ABC = \beta$, $\sphericalangle ACB = \gamma$. Let Q be the intersection of the tangents at B and C to the circumcircle of ABC. It now suffices to prove that G lies on PQ.

0 points.

Let T be the point on BC such that the line TA is tangent to the circumcircle of ABC at A. By the *tangent-chord theorem*, we have $\sphericalangle TAB = \gamma$, which implies $\sphericalangle ATB = \beta - \gamma$. Since $ZY \parallel TA$ and $\sphericalangle YDC = \sphericalangle ATB = \beta - \gamma$.

1 point.

By definion of Y, we have $|EY| = |DY|$. Since DE is a midline of ABC, we have $\sphericalangle EDC = \sphericalangle ABC = \beta$. Thus

$$\sphericalangle DEY = \sphericalangle EDY = \sphericalangle EDC - \sphericalangle YDC = \beta - (\beta - \gamma) = \gamma.$$

2 points.

Furthermore, note that $\sphericalangle BED + \sphericalangle DBE = \sphericalangle EDC = \beta$, and $\sphericalangle QBC = \alpha$, by the *tangent-chord theorem*. This implies

$$\sphericalangle QBE + \sphericalangle BEY = (\sphericalangle QBC + \sphericalangle DBE) + (\sphericalangle BED + \sphericalangle DEY) = \alpha + \beta + \gamma = 180^\circ$$

so $QB \parallel PE$. By analogous reasoning we conclude that $QC \parallel PF$.

2 points.

Since EF is a midline of ABC, we have that the sides EP, PF, EF of triangle PEF are parallel to the sides QB, QC, BC of triangle QBC, respectively. Those triangles are not congruent because $EF = \frac{BC}{2}$, so there exists a homothety which maps PEF to QBC. The centre of the homothety is the intersection of BE, CF, and PQ, which implies that G lies on PQ and we are done.

4 points.

Notes on marking:

- In the second solution, once we obtain $\sphericalangle DEY = \sphericalangle EDY = \gamma$, we can conclude that YE is tangent to the circumcircle of DEF and finish as in the first solution.
- The final **4 points** in either solution can only be awarded if the student correctly proves the other steps of the problem. Otherwise, a contestant can only obtain up to **2 points** for this part of the solution.
- No points are deducted if the student fails to argue that $\triangle PYZ$ and the triangle formed by the tangents through A, B and C to the circumcircle of ABC are not congruent.
- Analytic approaches are only awarded points if their results are correctly interpreted by geometric means.

Problem 3. Let $\mathbb{N}$ denote the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$x^2 - y^2 + 2y(f(x) + f(y))$$

is a square of an integer for all positive integers x and y.

(Ivan Novak)

First Solution. Throughout the solution, let $P(a, b)$ denote the assertion "$a^2 - b^2 + 2b(f(a) + f(b))$ is a perfect square".

Let p be a prime. Then $P(p, p)$ implies $4pf(p)$ is a perfect square, which implies $p \mid f(p)$.

1 point.

Let y be any positive integer, and let p be any prime. $P(p, y)$ implies $p^2 + 2yf(p) + 2yf(y) - y^2$ is a perfect square. Taking the assertion modulo p, it follows that $2yf(y) - y^2$ is a quadratic residue modulo p. It is a well known fact that if a positive integer is a quadratic residue modulo all primes, it must be a perfect square. We conclude that $2yf(y) - y^2$ is a perfect square for all $y \in \mathbb{N}$.

3 points.

Define $g(y)$ to be $\sqrt{2yf(y) - y^2}$.
$P(1, y)$ implies $1 - y^2 + 2yf(1) + 2yf(y) = g(y)^2 + 2yf(1) + 1$ is a perfect square, and since

$$g(y)^2 + 2yf(1) + 1 > g(y)^2,$$

we have a following chain of inequalities:

$$g(y)^2 + 2yf(1) + 1 \geqslant (g(y) + 1)^2 \implies 2yf(1) + 1 \geqslant 2g(y) + 1 \implies$$

$$yf(1) \geqslant g(y) \implies y^2 f(1)^2 \geqslant 2yf(y) - y^2 \implies \frac{f(1)^2 + 1}{2} y \geqslant f(y).$$

1 point.

Since $p \mid f(p)$ and $\frac{f(p)}{p} \leqslant \frac{f(1)^2+1}{2}$ for any prime p, it follows from Pigeonhole principle that there exists a positive integer a such that $f(p) = ap$ for infinitely many primes p.

2 points.

Let p be any prime such that $f(p) = ap$ and let n be any positive integer. $P(p, n)$ implies

$$p^2 - n^2 + 2nap + 2nf(n) = (p + na)^2 - n^2 a^2 - n^2 + 2nf(n)$$

is a perfect square.

However, this means that $2nf(n) - n^2 - n^2 a^2$ can be written as a difference of squares in infinitely many ways, which is only possible if it equals 0. Thus, $2nf(n) = n^2 a^2 + n^2$, or, equivalently, $f(n) = \frac{n(a^2+1)}{2}$ for all $n \in \mathbb{N}$. This also implies $\frac{a^2+1}{2} = a$, which gives us $a = 1$.
Therefore, $f(n) = n$ for all $n \in \mathbb{N}$. It can be easily checked that the identity function is indeed a solution.

3 points.

Second Solution. Similarly as in the first solution, we conclude that $p \mid f(p)$. Also, from $P(4p, 4p)$, we also conclude that $p \mid f(4p)$ for every odd prime p.

1 point.

Fix an odd prime number p, and let $A = \frac{f(p)}{p}$ and $B = \frac{f(4p)}{p}$.
Now, from $P(4p, p)$, we have that $15p^2 + 2p^2(A + B)$ is a square, which means $15 + 2(A + B)$ is also a square. Multiplying by 4 yields the fact that $60 + 8(A + B)$ is a square.
From $P(p, 4p)$, we have that $-15p^2 + 8p^2(A + B)$ is a square, which means $-15 + 8(A + B)$ is a square.
We then have

$$75 = 60 + 8(A + B) - (-15 + 8(A + B)).$$

Since there are finitely many ways to write 75 as a difference of two squares, we conclude that $A + B$ can take only finitely many different values as p ranges over all primes. The same is true for A.

4 points.

Therefore, by Pigeonhole principle, there exists a positive integer a such that $f(p) = ap$ for infinitely many primes p.

2 points.

The problem can now be finished the same way as in the first solution.

3 points.

Notes on marking:

- Both solutions follow a similar structure. In the first part, a constant a is found such that $f(p) = ap$ for infinitely many primes p, and in the second part the problem is completed using the first fact. The points from different solutions are not additive.
- In the first solution, it is not expected from the students to prove the key lemma about quadratic residues that is used; it suffices to state it. On the other hand, merely stating the lemma is not worth any points on its own.
- If a student solves the problem under the assumption that $f(p) = ap$ for infinitely many p, but they don't prove this fact, they can get at most **1 point** out of the last **3 points**.
- In the second solution, if a student discusses $P(1, 4)$ and $P(4, 1)$ similarly to how $P(p, 4p)$ and $P(4p, p)$ are discussed, and concludes that $f(1)$ and $f(4)$ can take on finitely many different values, they should get **1 point** out of **4** for that part of the solution.

Problem 4. Find all positive integers d for which there exist polynomials $P(x)$ and $Q(x)$ with real coefficients such that degree of P equals d and

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2.$$

(Ivan Novak)

First Solution.

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2 \tag{1}$$

Let P and Q be polynomials satisfying the conditions. Note that the degree of the left hand side in (1) the equality is $2d$, and the degree of the right hand side is $2 + 2\deg Q$, which implies $\deg Q = d - 1$.

Suppose that r is a real root of Q. Then $P(r)^2 + 1 = 0$, which is clearly impossible. We conclude that Q has no real roots. Since Q has real coefficients, we conclude that Q has even degree since its roots must come in conjugate pairs. Thus, d must be odd.

1 point.

Now we prove that for any odd d, there exist polynomials satisfying the conditions. Let $\mathbb{R}[x, \sqrt{x^2+1}]$ be the set of all functions of the form $A + B\sqrt{x^2+1}$, where A and B are polynomials with real coefficients. Note that each element of $\mathbb{R}[x, x^2 + 1]$ can be uniquely associated with a pair of polynomials (A, B). Consider a function $n : \mathbb{R}[x, \sqrt{x^2+1}] \to \mathbb{R}$ defined by

$$n(A + B\sqrt{x^2+1}) = A^2 - (x^2 + 1)B^2$$

for all real polynomials A and B. Note that the equality (1) is equivalent to the equality

$$n(P + \sqrt{x^2+1}Q) = -1.$$

1 point.

Note that

$$n((A+\sqrt{x^2+1}B)(C+\sqrt{x^2+1}D)) = n(AC+(x^2+1)BD+\sqrt{x^2+1}(AD+BC)) = (AC+(x^2+1)BD)^2-(x^2+1)(AD+BC)^2.$$

On the other hand,

$$n(A + \sqrt{x^2+1}B)n(C + \sqrt{x^2+1}D) = (A^2 - (x^2 + 1)B^2)(C^2 - (x^2 + 1)D^2).$$

It can easily be checked that the two expressions are equal. Hence, the function n is multiplicative.

1 point.

Note that $n(x + \sqrt{x^2+1}) = -1$.

1 point.

Then, using the multiplicative property, $n((x + \sqrt{x^2+1})^d) = -1$ as well. Let $(x + \sqrt{x^2+1})^d = P + \sqrt{x^2+1}Q$ for some polynomials P and Q. By binomial theorem, we have

$$P + \sqrt{x^2+1}Q = (x + \sqrt{x^2+1})^d = \sum_{j \text{ odd}} \binom{d}{j} x^{d-j}(x^2 + 1)^{\frac{d-j}{2}} + \sum_{j \text{ even}} \binom{d}{j} x^j (x^2 + 1)^{\frac{d-1-j}{2}} \sqrt{x^2+1}.$$

It's now easy to see that P has degree d, since it is a sum of polynomials which have degree d and positive leading coefficients, and P and Q satisfy the starting equality. Thus, all odd positive integers are solutions.

6 points.

Second Solution.

$$P(x)^2 + 1 = (x^2 + 1)Q(x)^2 \tag{2}$$

Similarly as in the first solution, we conclude that d needs to be odd.

1 point.

Let us now prove that for every odd d such polynomials $P(x)$ and $Q(x)$ exist. Fix an odd positive integer d. Observing the roots of polynomials $P(x)$ and $Q(x)$, we can easily see from (2) that $P(x)$ and $Q(x)$ don't have a common root. Differentiating (2), we get :

$$P'(x)P(x) = x \cdot Q(x)^2 + (x^2 + 1) \cdot Q'(x)Q(x) = Q(x)\left(x \cdot Q(x) + (x^2 + 1) \cdot Q'(x)\right). \tag{3}$$

Since $P(x)$ and $Q(x)$ don't have common roots, from (3) we conclude that $Q(x)$ must divide $P'(x)$. Since they have the same degree, there must exist a real number u such that $uP'(x) = Q(x)$.

1 point.

Comparing coefficients in (2), we get that u must be $\frac{1}{d}$ or $-\frac{1}{d}$.

1 point.

We'll take $u = \frac{1}{d}$. Plugging in $Q(x) = P'(x)/d$ in (3), we get

$$P(x) = \frac{1}{d}\left(\frac{1}{d}x \cdot P'(x) + \frac{1}{d}(x^2+1)P''(x)\right). \tag{4}$$

We will now find all polynomials P of degree d which satisfy (4). Note that, by multiplying both sides with $P'(x)$ and integrating, each of these polynomials satisfies the equation $P(x)^2 + C = (x^2+1)\frac{(P'(x))^2}{d^2}$ for some $C \in \mathbb{R}$.

Denote $P(x) = \sum_{i=0}^{d} a_i x^i$. Then $P'(x) = \sum_{i=1}^{d} i a_i x^{i-1}$ and $P''(x) = \sum_{i=2}^{d} i(i-1)a_i x^{i-2}$. Writing out the coefficients in (4), we get

$$\sum_{i=0}^{d} a_i x^i = \frac{1}{d^2}\left(\sum_{i=1}^{d} i^2 a_i x^i + \sum_{i=2}^{d} i(i-1)a_i x^{i-2}\right)$$

Comparing the coefficients of x^k for all k on the left hand side and the right hand side of the above equation for $k \geqslant 0$, we get:

$$a_{d-1} = \frac{1}{d^2}\left((d-1)^2 a_{d-1}\right) \quad \text{which implies} \quad a_{d-1} = 0,$$

and also

$$a_k = \frac{1}{d^2}\left(k^2 a_k + (k+2)(k+1)a_{k+2}\right) \quad \text{which can be rewritten as}$$

$$a_{k+2} = \frac{d^2 - k^2}{(k+2)(k+1)} \cdot a_k \quad \text{for all} \quad 0 \leqslant k \leqslant d-2.$$

1 point.

From here, we now have that $a_k = 0$ for all even $0 \leqslant k \leqslant d-2$ and that for all odd $0 \leqslant k \leqslant d-2$, we have $a_k = q_k a_1$ for some nonzero real coefficient q_k which is uniquely determined by the above recursion.

1 point.

It's easy to see that any such choice of coefficients $(a_k)_k$ with $a_1 \neq 0$ gives a solution to (4) which has degree d.

As we've already said, any solution to (4) is a solution to $P(x)^2 + C = (x^2+1)\frac{(P'(x))^2}{d^2}$ for some $C \in \mathbb{R}$. Considering the coefficient alongside x^0 in both sides and noting $a_0 = 0$, we get $C = a_1^2/d^2$. Thus, taking a solution with $a_1 = d$, we get the solution to $P(x)^2 + 1 = (x^2+1)\frac{(P'(x))^2}{d^2}$, which proves that every odd d is a solution to the problem.

5 points.

Third Solution.

$$P(x)^2 + 1 = (x^2+1)Q(x)^2 \tag{5}$$

Similarly as in the first solution, we conclude that d needs to be odd.

1 point.

Note that $d = 1$ is a solution, taking $P(x) = x$ and $Q(x) = 1$. Henceforth assume $d \geqslant 3$.

From $x^2+1 \mid P(x)^2+1$ we get $x^2+1 \mid P(x)^2 - x^2 \implies x^2+1 \mid (P(x)-x)(P(x)+x)$. It is not hard to see that the irreducible polynomial x^2+1 divides exactly one of the two factors. We can replace P by $-P$, so without loss of generality it is safe to assume that $P(x) = A(x)(x^2+1) + x$ for some real polynomial A. If we put this in (5), we obtain

$$(A(x) \cdot x + 1)^2 + A(x)^2 = Q(x)^2.$$

We will find real polynomials α and β such that $A(x) = 2\alpha(x)\beta(x)$, $Q(x) = \alpha(x)^2 + \beta(x)^2$, $xA(x) + 1 = \alpha(x)^2 - \beta(x)^2$ and $\deg\alpha + \deg\beta = d-2$. Note that then $A(x)$ and $Q(x)$ satisfy the conditions due to the identities for Pythagorean triples.

2 points.

We thus need to find solutions to the equation

$$2x\alpha(x)\beta(x) + 1 = \alpha(x)^2 - \beta(x)^2 \tag{6}$$

where $\alpha, \beta \in \mathbb{R}[x]$ are polynomials with real coefficients. Notice that $(\alpha, \beta) = (1, -2x)$ is a pair of solutions.

1 point.

If we look at (6) as a quadratic equation in α we have

$$\alpha^2 - \alpha \cdot 2x\beta + 1 - \beta^2 = 0.$$

Roots α_1, α_2 must then satisfy $\alpha_1 + \alpha_2 = 2x\beta$. It is now easily verified that if (α, β) is a pair of polynomials which satisfy (6), then $(\beta, 2x\beta - \alpha)$ is another such pair.

1 point.

Thus starting with solution $(\alpha_0, \beta_0) = (1, -2x)$, we can recursively generate a sequence of solutions

$$(\alpha_{i+1}, \beta_{i+1}) = (\beta_i, 2x\beta_i - \alpha_i).$$

The degrees of $(\alpha_i, \beta_i)_{i \geqslant 0}$ now follow the pattern

$$(0, 1), (1, 2), (2, 3), (3, 4) \dots$$

More precisely, $\deg \alpha_i = i$, $\deg \beta_i = i + 1$ for all $i \geqslant 0$.
But then, if $d = 2i+1$ for some $i \geqslant 0$, the pair (α_i, β_i) gives a pair $(A, Q) = (2\alpha\beta, \alpha^2+\beta^2)$ such that $(xA(x)+1)^2+A(x)^2 = Q(x)^2$ and $\deg A = i + (i-1) = d - 2$. Then, taking $P(x) = (x^2 + 1)A(x) + x$ yields a pair (P, Q) satisfying the original equation such that $\deg P = d$. We conclude that every odd d is a solution.

5 points.

Notes on marking:

- Points from different marking schemes are not additive.