10TH EUROPEAN MATHEMATICAL CUP

11th December 2021 - 19th December 2021

Junior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problems and Solutions

Problem 1. We say that a quadruple of nonnegative real numbers (a, b, c, d) is *balanced* if

$$a + b + c + d = a^2 + b^2 + c^2 + d^2.$$

Find all positive real numbers x such that

$$(x - a)(x - b)(x - c)(x - d) \geqslant 0$$

for every balanced quadruple (a, b, c, d).

(Ivan Novak)

First Solution. We'll call any $x \in \langle 0, \infty\rangle$ satisfying the problem's condition *great*. Let (a, b, c, d) be a balanced quadruple. Without loss of generality let $a \geqslant b \geqslant c \geqslant d$. We can rewrite the equation $a^2 + b^2 + c^2 + d^2 = a + b + c + d$ as

$$\left(a - \frac{1}{2}\right)^2 + \left(b - \frac{1}{2}\right)^2 + \left(c - \frac{1}{2}\right)^2 + \left(d - \frac{1}{2}\right)^2 = 1,$$

which implies $(a - \frac{1}{2})^2 \leqslant 1$, meaning that $a \leqslant \frac{3}{2}$.

6 points.

If we take $x \geqslant \frac{3}{2}$, the values of $x - a,\ x - b,\ x - c$ and $x - d$ are all nonnegative. Thus, any $x \geqslant \frac{3}{2}$ is great.

1 point.

If we take $(a, b, c, d) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, then for any $x \in \langle \frac{1}{2}, \frac{3}{2}\rangle$ we have $(x - a)(x - b)(x - c)(x - d) < 0$. Thus, no $x \in \langle \frac{1}{2}, \frac{3}{2}\rangle$ is great.

2 points.

If we take $(a, b, c, d) = (1, 0, 0, 0)$, then for any $x \in \langle 0, 1\rangle$ we have $(x - a)(x - b)(x - c)(x - d) < 0$. Thus, no $x \in \langle 0, 1\rangle$ is great.

1 point.

We conclude that a number x is great if and only if $x \geqslant \frac{3}{2}$.

Second Solution. Here we present another way to conclude that all $x \geqslant \frac{3}{2}$ satisfy the condition. As in the first solution, we call any $x \in \langle 0, \infty\rangle$ which satisfies the problem's condition *great*, and without loss of generality let (a, b, c, d) be a balanced quadruple satisfying $a \geqslant b \geqslant c \geqslant d$. We notice that for all $y \in \mathbb{R}$ we have

$$\left(y - \frac{1}{2}\right)^2 \geqslant 0 \implies y^2 \geqslant y - \frac{1}{4}.$$

Applying this inequality to b, c and d separately and summing the inequalities we get the following:

$$\begin{cases} b^2 \geqslant b - \frac{1}{4} \\ c^2 \geqslant c - \frac{1}{4} \\ d^2 \geqslant d - \frac{1}{4} \end{cases} \implies b^2 + c^2 + d^2 \geqslant b + c + d - \frac{3}{4}.$$

3 points.

Using the equality $b^2 + c^2 + d^2 = a + b + c + d - a^2$ transforms the inequality above into

$$a \geqslant a^2 - \frac{3}{4} \implies 1 \geqslant \left(a - \frac{1}{2}\right)^2,$$

which implies $a \leqslant \frac{3}{2}$,

3 points.

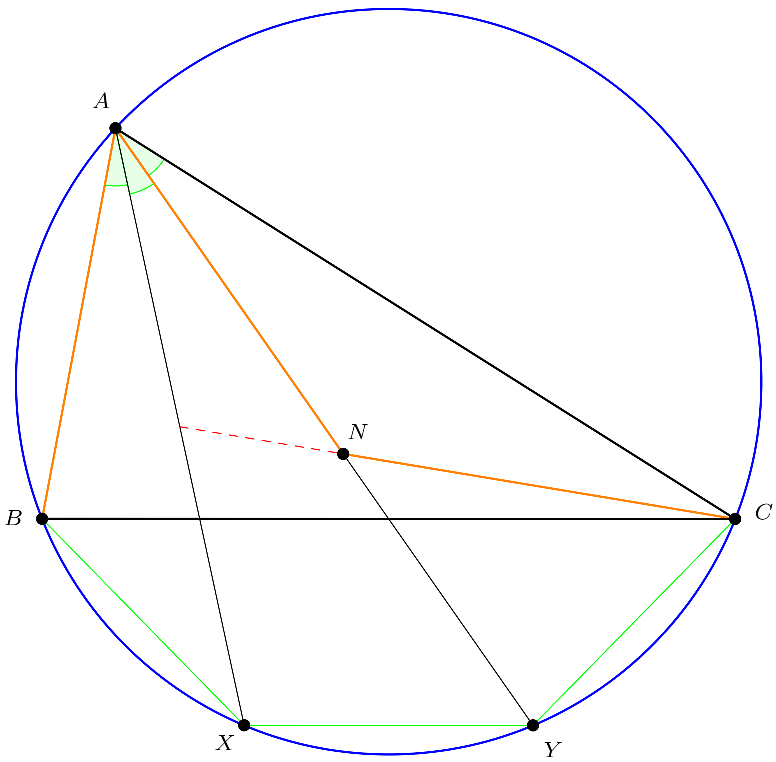
meaning that all $x \geqslant \frac{3}{2}$ are great.

1 point.

The rest of the solution is the same as the previous solution.

Problem 2. Let ABC be an acute-angled triangle such that $|AB| < |AC|$. Let X and Y be points on the minor arc BC of the circumcircle of ABC such that $|BX| = |XY| = |YC|$. Suppose that there exists a point N on the segment $\overline{AY}$ such that $|AB| = |AN| = |NC|$. Prove that the line NC passes through the midpoint of the segment $\overline{AX}$.

(Ivan Novak)



First Solution. Let the line CN intersect the circumcircle of ABC at $T \neq C$.

Since $|BX| = |XY| = |YC|$, we have $\sphericalangle BAX = \sphericalangle XAY = \sphericalangle YAC$. Denote that angle by φ. Furthermore, since $|AN| = |NC|$, we have $\sphericalangle NAC = \sphericalangle NCA = \varphi$, which means that $|AT| = |CY|$.

1 point.

Furthermore, from $|AB| = |AN|$ and $\sphericalangle BAX = \sphericalangle XAY$ it follows that AX is the perpendicular bisector of $\overline{BN}$, so $|XN| = |BX|$. Since $|BX| = |CY| = |AT|$, we have $|XN| = |AT|$.

3 points.

Now $ATBX$ is an isosceles trapezoid because $|AT| = |BX|$ and because it's cyclic, which means that $|AB| = |TX|$, which, combined with the fact that $|AB| = |AN|$, yields $|TX| = |AN|$.

3 points.

Therefore, from $|TX| = |AN|$ and $|XN| = |AT|$ we have that triangles ATX and ANX are congruent, as well as triangles ATN and XTN. Therefore, $\sphericalangle ATX = \sphericalangle ANX$ and $\sphericalangle TAN = \sphericalangle TXN$, which means that $ATXN$ is a parallelogram. Now, as NT and AX are diagonals of a parallelogram, NT passes through the midpoint of $\overline{AX}$, which proves the claim.

3 points.

Second Solution. Let l be the line parallel to AB through C, and let P be the intersection of l and AY. Let α denote the angle $\angle BAC$.

From $|AN| = |NC|$ it follows that $\angle ABX = \angle XAN = \angle NAC = \angle NCA = \frac{\alpha}{3}$. Then it's easy to see that $\angle CNP = \angle NAC + \angle NCA = \frac{2\alpha}{3}$.

1 point.

Similarly, $\angle NPC = \angle PAB = \frac{2\alpha}{3}$ by definition of P.

2 points.

Thus, the triangle CNP is isosceles. Therefore, we conclude that $|CN| = |PC|$.

1 point.

However, $|CN| = |AN| = |AB|$, which implies $|AB| = |PC|$. Since AB and PC are parallel, this means that $ABPC$ is a parallelogram.

2 points.

This implies that AY is the A-median of triangle ABC. Since AX is isogonal to AY with respect to $\sphericalangle BAC$, we conclude that AX is the A-symmedian in the triangle ABC.

1 point.

From a well-known lemma, it now follows that CB is the C-symmedian in the triangle AXC. Note that

$$\angle BCX = \angle BAX = \angle NAC = \angle NCA.$$

We now see that CN and CB are isogonal with respect to $\angle ACX$. Hence, CN is the C-median of triangle ACX and we are done.

3 points.

Third Solution. Denote the angles of ABC by α, β and γ in a standard way, so that $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

Note that $\angle BCX = \frac{\alpha}{3}$ and $\angle WCA = \angle NCA = \angle NAC = \frac{\alpha}{3}$ since NCA is isosceles. Thus, $\angle WCX = \angle BCA = \gamma$. Also, note that $\angle WAC = \frac{2\alpha}{3}$.

From sine law in triangle ANC, we have $\frac{|AN|}{\sin\frac{\alpha}{3}} = \frac{|AC|}{\sin\frac{2\alpha}{3}}$. Using the fact that $\frac{|AB|}{|AC|} = \frac{\sin\gamma}{\sin\beta}$ and $|AB| = |AN|$, we get the equality

$$\sin\gamma \sin\frac{2\alpha}{3} = \sin\beta \sin\frac{\alpha}{3}. \quad (1)$$

3 points.

Let W denote the intersection of AX and NC. From sine law in triangle AWC, we have

$$\frac{|AW|}{|WC|} = \frac{\sin\frac{\alpha}{3}}{\sin\frac{2\alpha}{3}}.$$

1 point.

From sine law in WXC, we have

$$\frac{|WX|}{|WC|} = \frac{\sin\gamma}{\sin\beta}.$$

1 point.

From these two equalities we get

$$\frac{|AW|}{|WX|} = \frac{\sin\beta\sin\frac{\alpha}{3}}{\sin\gamma\sin\frac{2\alpha}{3}},$$

which equals 1 by (1). Thus, $|AW| = |WX|$.

5 points.

Notes on marking:

- In the second solution, proving that it suffices to prove that AY is the A-median of ABC (last **4 points**) is worth at most **2 points** if a student doesn't prove the parts before that.

Problem 3. Let ℓ be a positive integer. We say that a positive integer k is *nice* if $k! + \ell$ is a square of an integer. Prove that for every positive integer $n \geqslant \ell$, the set $\{1, 2, \ldots, n^2\}$ contains at most $n^2 - n + \ell$ nice integers.

(Theo Lenoir)

Solution. We claim that for every $k \geqslant \ell + 1$, at most one number among $k^2 - 1$ and k^2 is nice.

1 point.

Suppose for the sake of contradiction that both $k^2 - 1$ and k^2 are nice for some $k \geqslant \ell + 1$.
Let $u = \sqrt{(k^2-1)! + \ell}$ and $v = \sqrt{(k^2)! + \ell}$. Then

$$v^2 - \ell = (k^2)! = k^2(u^2 - \ell),$$

which can be rearranged into

$$(ku)^2 - v^2 = (k^2 - 1)\ell. \quad (1)$$

2 points.

Note that this implies $ku > v$ and, furthermore,

$$(ku)^2 - v^2 = (ku - v)(ku + v) \geqslant ku + v > ku > k\sqrt{(k^2-1)!} = \sqrt{(k^2)!}$$

Furthermore, we have the following bounds:

$$(k^2)! \geqslant k^2(k^2-1)(k^2-2) > k^2(k^2-1)(k-1)^2 > (k^2-1)^2(k-1)^2 \geqslant (k^2-1)^2\ell^2,$$

where we used the fact that $k^2 - 2 > (k-1)^2 = k^2 - 2k + 1$ for $k \geqslant 2$ and the assumption $\ell \leqslant k - 1$. But this implies that the left hand side in (1) is greater than the right hand side, which is a contradiction.

7 points.

Thus, there is at least one integer which is not good among $\{k^2 - 1, k^2\}$ for every $k \in \{\ell + 1, \ldots, n\}$, which means there are at least $n - \ell$ integers which aren't good. Thus, the claim is proven.

Partial solution. This is a sketch of a partial solution using analytic number theory. This is not a solution to the original problem, but it provides a better asymptotic bound on the number of nice integers. This solution is worth **5 points** in total.
We first solve the case where ℓ is not a perfect square. Let p be a prime such that $\nu_p(\ell)$ is odd. Then for every $k \geqslant 2p$, we have $\nu_p(k! + \ell) = \nu_p(\ell)$, which is odd. Hence, every $k \geqslant 2p$ is not nice, so there are at most 2ℓ nice numbers and $2\ell \leqslant n^2 - n + \ell$ for $\ell \geqslant 2$.

1 point.

Now consider the case when ℓ is a square. Then $\ell + 1$ is not a perfect square. Note that $(p-2)! \equiv 1 \pmod p$ for every prime number p, and thus $(p-2)! + \ell \equiv \ell + 1 \pmod p$. If we pick p to be a prime such that $\ell + 1$ is not a quadratic residue modulo p, we conclude that $(p-2)! + \ell$ is not a square.

1 point.

Note that, by quadratic reciprocity, $\ell + 1$ being a quadratic residue modulo p for $p > \ell + 1$ depends only on the remainder of p modulo $8(\ell + 1)$, and since $\ell + 1$ is not a square, there must exist a class of residues modulo $8(\ell + 1)$ such that $\ell + 1$ is not a quadratic residue modulo primes from that class.

1 point.

By Dirichlet's theorem on arithmetic progressions, the number of primes less than some N which are from a given class of residues modulo $8(\ell + 1)$ is asymptotically

$$\frac{1}{\varphi(8(\ell+1))}\pi(N),$$

where $\pi(N)$ is the number of primes less than N and φ is the Euler's Totient function. By Prime number theorem, $\pi(N)$ is asymptotically $N/\log(N)$. Hence, for large n, the number of integers less than n^2 which are not nice is at least

$$c \cdot \frac{n^2}{\log(n^2)},$$

where $c > 0$ is some constant. For n large enough, this is obviously bigger than $n - \ell$.

2 points.

Notes on marking:

- In the second solution, if a contestant isn't rigorous enough with the bounds in the end, they shouldn't get more than **1 point** for the last part.
- Points from the second solution are not additive with the points from the first solution.

Problem 4. Let n be a positive integer. Morgane has coloured the integers $1, 2, \ldots, n$. Each of them is coloured in exactly one colour. It turned out that for all positive integers a and b such that $a < b$ and $a + b \leqslant n$, at least two of the integers among a, b and $a + b$ are of the same colour. Prove that there exists a colour that has been used for at least $2n/5$ integers.

(Vincent Juge)

First Solution. Throughout the solution, instead of colourings, we will consider partitions, and 'being coloured in the same colour' will be interpreted as 'being in the same element of a partition', and the colours will be interpreted as the blocks of partitions.

Let A denote the first colour that appears, i.e. the block that contains 1. Also, let B denote the block which contains the first integer not in A.

We shall prove that either A or B has at least $2n/5$ elements.

Let C be the union of all blocks other than A and B, and let b be the smallest element of B.

Lemma 1. *For any x, if $x \in C$ then $x - 1 \in A$ and either $x + 1 \in A$ or $x = n$.*

Proof. We'll actually prove a stronger claim: If $x \in C$, then $x-1, x-2, \ldots, x-(b-1) \in A$ and $x+1, x+2, \ldots, x+(b-1) \in A$ if they're not greater than n.

1 point.

For the sake of contradiction, consider the least x for which this claim doesn't hold. But then $x - j \notin C$ for any $j < b$ since otherwise $x - j$ would be the least counterexample since $(x - j) + j \in C$. Since $j \in A$ for $j < b$, we conclude that $x - j \in A$ for all $j < b$.

Now, for any $i < b$, we have $b \in B$ and $x + i - b \in A$, which implies $x + i \in A \cup B$. We also have $x \in C$ and $i \in A$, which implies $x + i \in C \cup A$. Hence, $x + i \in A$. But then $x + 1, \ldots, x + b - 1$ are all in A, a contradiction.

We conclude that the stronger claim holds for every x. Thus, the lemma is proven.

1 point.

□

Lemma 2. There do not exist x and y such that $x \in B$, $x + 1 \in B$, $y \in C$, $y + 2 \in C$.

Proof. Assume that there exist such x and y and assume that such x is minimal. Note that then $x - 1 \notin B$ and $1 \in A$, so $x - 1 \in A$.

We now distinguish two cases, depending on whether $x > y$ or not.

- If $x > y$, consider the integer $r = x - y = (x + 1) - (y + 1)$. Since $x \in B$ and $y \in C$, $r \in B \cup C$. Since $x + 1 \in B$ and $y + 1 \in A$, $r \in B \cup A$. Hence, $r \in B$. Similarly, consider $r + 1 = (x + 1) - y = x - (y - 1)$. Since $x + 1 \in B$ and $y \in C$, $r + 1 \in B \cup C$. Since $x \in B$ and $y - 1 \in A$, $r + 1 \in A \cup B$. Hence, $r + 1 \in B$.
- If $x < y$, consider the integer $r = (y + 1) - x = (y + 2) - (x + 1)$. Since $y + 1 \in A$ and $x \in B$, $r \in A \cup B$. Since $y + 2 \in C$ and $x + 1 \in B$, e $r \in C \cup B$. We conclude that $r \in B$.

  However, $y \in C$ and $y + 1 - x \in B$ implies that the integer $x - 1 = y - (y + 1 - x)$ is either in B or C, but we've already proven that $x - 1 \in A$. Thus, we've reached a contradiction.

3 points.

However, since $r = x - y < x$, this contradicts the minimality of x and we've reached a contradiction again.

3 points.

□

Now we've proved that there either doesn't exist $x \in B$ such that $x + 1 \in B$, or that there doesn't exist $y \in C$ such that $y + 2 \in C$.

In the first case, for every $x \in B$, we have $x - 1 \notin B$ and $1 \in A$. Hence, $x - 1 \in A$. But then $|A| \geqslant n/2$, since $x \mapsto x - 1$ is an injective function from $B \cup C$ to A.

1 point.

In the second case, for every $y \in C$, both $y + 1$ and $y - 1$ are either in A or greater than n, while $y + 2$ and $y - 2$ are not in A. Thus, $y \mapsto y - 1$ and $y \mapsto y + 1$ are injective functions from C to A and $A \cup \{n + 1\}$ respectively, and their images are disjoint. Additionally noting that $1 \in A$ and $1 \neq y - 1, y + 1$ for any $y \in C$, we conclude that $|C| \leqslant \frac{|A|}{2}$. But then $n = |A| + |B| + |C| \leqslant \frac{3|A|+2|B|}{2}$, so either $|A|$ or $|B|$ must be greater than or equal to $\frac{2n}{5}$.

1 point.

Second Solution. We give an alternative proof of **Lemma 1**.

Assume for the sake of contradiction that there are two consecutive integers x and $x+1$ such that both are in C. Let x be the smallest integer with that property. Consider the integer $(x+1)-b = x-(b-1)$. Since $x+1 \in C$ and $b \in B$, we have either $x+1-b \in C$ or $x+1-b \in B$. Since $x \in C$ and $b-1 \in A$, we have either $x+1-b \in C$ or $x+1-b \in A$. We conclude that $x+1-b \in C$.

Furthermore, considering $x-b = (x+1-b)-1$, we have $x \in C$, $b \in B$, $x+1-b \in C$ and $1 \in A$. We conclude that $x-b \in C$, but then $x-b$ and $x+1-b$ are both in C, contradicting the minimality of x. We conclude that there are no consecutive integers in C.

1 point.

Now, for all $x \in C$, the integers $x-1$ and $x+1$ must either be equal to $n+1$ or belong to A, since $1 \in A$, $x-1 \notin C$ and $x+1 \notin C$.

1 point.

□

We also give a slightly different proof of **Lemma 2.**

Proof. We first prove that if x and $x+1$ are both in B and are greater than $y \in C$, then $x-y+1$ and $x-y$ are both in B. Note that $y+1 \in A$ and $y-1 \in A$ by **Lemma 1.**

Note that $x-y = (x+1)-(y+1)$, so $x-y \in B \cup C$ and $x-y \in B \cup A$, which implies $x-y \in B$. Similarly, considering $(x+1)-y = x-(y-1)$, we can conclude that $x-y+1 \in B$.

2 points.

To now prove **Lemma 2**, it suffices to consider the case $x < y$, where x and y are from the statement of the lemma.

1 point.

We deal with that case in the same way as in the first solution.

3 points.

□

The remainder of the solution is the same.

2 points.

Notes on marking:

- Lemma 2 is worth **6 points**. If a contestant states Lemma 2 and they don't prove any of its two subcases, they should get **1 point** for Lemma 2.
- In the second solution, proving the first part of Lemma 2 is worth less than the case $x > y$ in the first solution, because one can prove this part without stating Lemma 2. Proving this part in the context of Lemma 2 is worth **3 points**, and without the context of Lemma 2 it's worth **2 points**.