9TH European Mathematical Cup

12th December 2020 - 20th December 2020

Senior Category

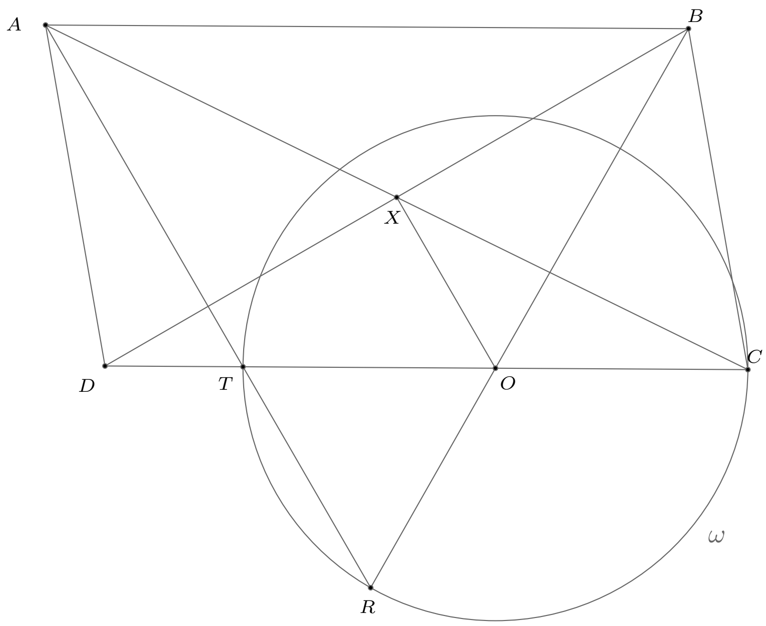
Problems and Solutions

Problem 1. Let $ABCD$ be a parallelogram in which $|AB| > |BC|$. Let O be a point on the line CD such that $|OB| = |OD|$. Let ω be a circle with center O and radius $|OC|$. If T is the second intersection of ω and CD, prove that AT, BO and ω are concurrent.

First Solution. Let R denote the intersection od ω and line BO such that O is located between B and R. We will prove that A, T and R are collinear.

0 points.

Let X be the intersection of the diagonals of $ABCD$.



We know that X is the midpoint of $\overline{AC}$ and O is the midpoint of $\overline{TC}$ so we conclude that $XO||AT$.

2 points.

X is also the midpoint of $\overline{BD}$ so, since triangle OBD is isosceles, $OX \perp BD$.

2 points.

This means that $AT \perp BD$.

1 point.

Now because of $|DO| = |BO|$ we have

$$\angle DOR = 2\angle ODB$$

and because $|OT| = |OR|$ we have

$$\angle OTR = 90^o - \angle ODB$$

2 points.

Finally we have

$$\angle ATD = 90^o - \angle BDC = \angle OTR$$

and so A, T and R are collinear as desired.

3 points.

Second Solution. Define R as the intersection of the ray BO with ω such that O is between B and R. We will prove that A, T and R are collinear.

0 points.

Since $|BO| = |DO|$ and $|OR| = |OC|$, we have:

$$|BR| = |BO| + |OR| = |DO| + |OC| = |CD| = |BA|.$$

Therefore, triangle BRA is isosceles.

5 points.

Now, due to the triangles TOR and BRA being isosceles, we have:

$$|\angle BRA| = \frac{180^\circ - |\angle RBA|}{2}$$

and

$$\frac{180^\circ - |\angle ROT|}{2} = |\angle ORT| = |\angle BRT|$$

.

2 points.

Finally, since $|\angle RBA| = |\angle TOR|$, we have

$$|\angle BRA| = |\angle BRT|$$

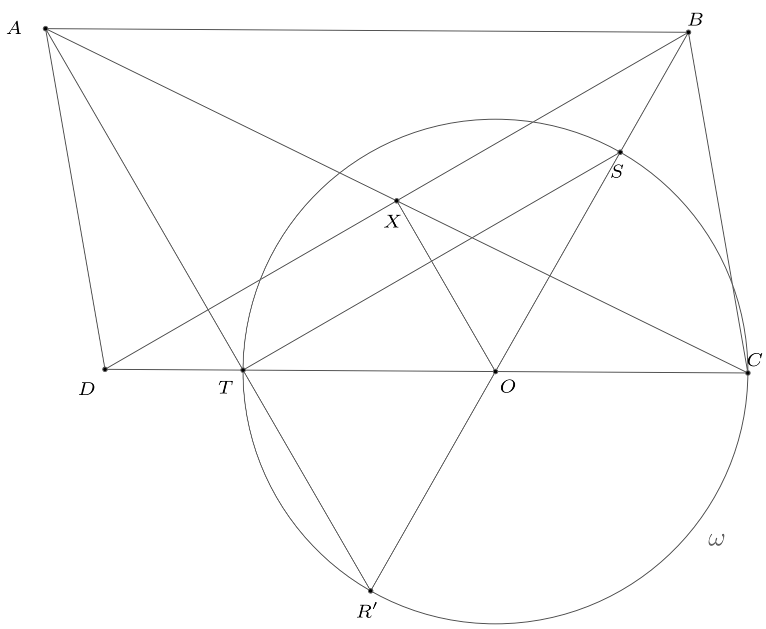
, so R, T and A are collinear, which proves the claim.

3 points.

Third Solution. Let R' be the intersection of line AT and ω different from T. We will prove that points B, O and R' are collinear.

0 points.

Let X be the intersection of the diagonals of the parallelogram $ABCD$.



Now as in the first solution we conclude that $XO||AT$ and $OX \perp BD$, which leads to AT being perpendicular to BD.

5 points.

Let S be the intersection of ω and ray OB. Since triangles ODB and OTS are isosceles with $\angle DOB = \angle TOS$, these triangles are similar, which means that $TS||BD$.

2 points.

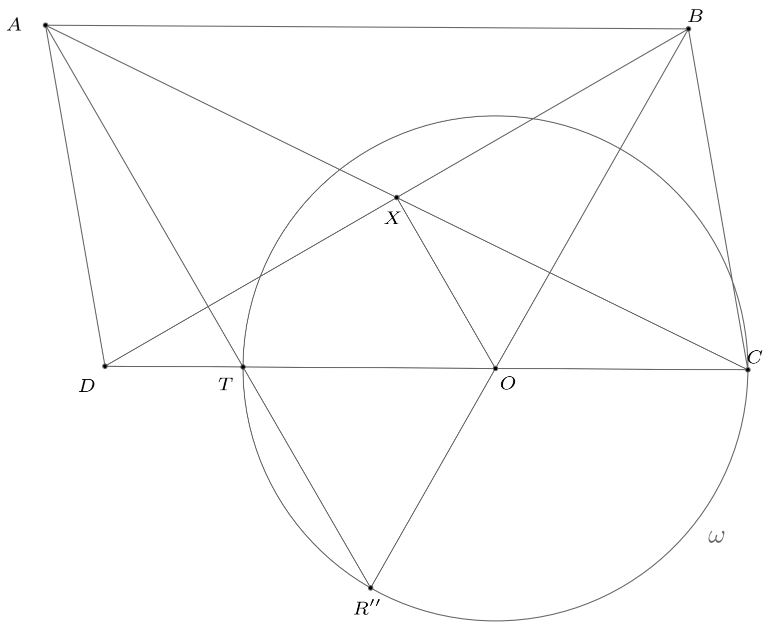
From this it follows that $ST \perp AT$, i.e. $\angle STR' = 90^\circ$. This means that $\overline{SR'}$ is the diameter of ω, and as we know that B, S and O are collinear, we conclude that B, O and R' are collinear.

3 points.

Fourth Solution. Let R'' be the intersection of lines AT and BO. We will show that R'' lies on the circle ω.

0 points.

Let X be the intersection of the diagonals of the parallelogram $ABCD$.



As in the first solution we conclude that $XO||AT$, $OX \perp BD$ and $AT \perp BD$.

5 points.

Denote $\angle TR''O = \alpha$. Since $AT \perp BD$, we have

$$\angle OBD = 90^\circ - \alpha.$$

Now, due to the triangle ODB being isosceles, we have

$$\angle ODB = 90^\circ - \alpha.$$

2 points.

Using again the fact that $AT \perp BD$, it follows that

$$\angle R''TO = \angle ATD = \alpha.$$

We can now conclude that $|OT| = |OR''|$, which proves the claim.

3 points.

Notes on marking:

- Points from different solutions are not additive. Student's score should be the maximum of points scored over all solutions.
- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a complete solution can be awarded.

Problem 2. Let n and k be positive integers. An n-tuple $(a_1, a_2, \ldots, a_n)$ is called a *permutation* if every number from the set $\{1, 2, \ldots, n\}$ occurs in it exactly once. For a permutation $(p_1, p_2, \ldots, p_n)$, we define its k-*mutation* to be the n-tuple

$$(p_1 + p_{1+k}, p_2 + p_{2+k}, \ldots, p_n + p_{n+k}),$$

where indices are taken modulo n. Find all pairs (n, k) such that every two distinct permutations have distinct k-mutations.

Remark: For example, when $(n, k) = (4, 2)$, the 2-mutation of $(1, 2, 4, 3)$ is $(1+4, 2+3, 4+1, 3+2) = (5, 5, 5, 5)$.

(Borna Šimić)

First Solution. Let f denote the function that, when given a permutation, returns its k-mutation.
Let $M(a, b)$ denote the greatest common divisor of a and b.
The answer is all (n, k) such that $n/M(n, k)$ is odd.

Suppose that $n/M(n, k)$ is odd.
Consider permutations p, q such that $f(p) = f(q)$. Suppose for the sake of contradiction that there exists some $t \leqslant n$ such that $p_t > q_t$. We have:

$$p_t + p_{t+k} = q_t + q_{t+k}$$

so we must have $p_{t+k} < q_{t+k}$, and $p_{t+2k} > q_{t+2k}$. Inductively, we obtain $p_{t+dk} < q_{t+dk}$ for all odd d (where the indices are taken modulo n).

2 points.

However, $n/M(n, k)$ is odd and we have $p_{t+nk/M(n,k)} = p_t$ and $q_{t+nk/M(n,k)} = q_t$. However, then $p_t < q_t$, which is a contradiction. Therefore, $p = q$, which proves that all (n, k) for which $n/M(n, k)$ is odd are solutions.

3 points.

We will now show that when $n/M(n, k)$ is even, there exist distinct permutations p, q such that $f(p) = f(q)$.
Firstly, fix n, and for $(n, k) = (2m, 1)$ for some $m \in \mathbb{N}$ take:

$$p^1 = (1, 2, 3, \ldots 2m-1, 2m)$$
$$q^1 = (2, 1, 4, \ldots 2m, 2m-1)$$

It's easy to see that $f(p^1) = f(q^1)$.

1 point.

Now, if $k = 2u - 1$ for some $u \geqslant 1$ such that $M(n, 2u-1) = 1$, define permutations p^u, q^u by taking

$$p^u_{(m-1)k+1} = p^1_m \text{ and } q^u_{(i-1)k+1} = q^1_i,$$

where indices are taken modulo n. (For example, for p^1 and p^u, $p^u_1 = p^1_1 = 1, p^u_{k+1} = p^1_2 = 2$ and so on).
As k and n are relatively prime, p^u and q^u are well defined, because the map $x \mapsto (x-1)k+1$ is a bijection on the set of residues modulo n. Furthermore, it's easy to see that $f(p^u) = f(q^u)$ holds, because $f(p^1) = f(q^1)$ holds.

2 points.

Finally, a construction for $(n, 2u-1)$ can be expanded to a construction for $(ln, l(2u-1))$, by defining $p(lj) = p^u(j)$ and $q(lj) = q^u(j)$ for every j, and setting $p(x) = q(x)$ for x which are not divisible by l (it is not important how p and q are defined on the set of numbers not divisible by l, it's only important that they are equal on this set). Since $f(p^u) = f(q^u)$, we conclude that $f(p) = f(q)$ also holds.

Since any pair of positive integers (n, k) for which $n/M(n, k)$ is even can be written in this form, we've proved the claim.

2 points.

Second Solution. Let the notation be the same as in the first solution. Let d be an odd positive integer. Consider some permutations p, q such that $f(p) = f(q)$. This gives us the following sequence of equations for $i = 1, 2, \ldots, n$:

$$p_i + p_{i+k} = q_i + q_{i+k} \tag{1}$$
$$p_{i+k} + p_{i+2k} = q_{i+k} + q_{i+2k} \tag{2}$$
$$\vdots = \vdots$$
$$p_{i+(d-2)k} + p_{i+(d-1)k} = q_{i+(d-2)k} + q_{i+(d-1)k} \tag{$d-1$}$$
$$p_{i+(d-1)k} + p_{i+dk} = q_{i+(d-1)k} + q_{i+dk} \tag{d}$$

We telescope the equations: $(d-1)-(d-2)+(d-3)-\ldots+\ldots-(1)$. Since d is odd, we obtain:

$$p_{i+(d-1)k} - p_i = q_{i+(d-1)k} - q_i$$

We subtract that from (d) and obtain: $p_{i+dk} + p_i = q_{i+dk} + q_i$.

2 points.

Since this equality holds for every odd d, it also holds for $n/M(n,k)$. Since $p_{i+nk/M(n,k)} = p_i$ and $q_{i+nk/M(n,k)} = q_i$, we conclude that $2p_i = 2q_i$ for all i. Therefore, $p = q$.

3 points.

The case where $n/M(n,k)$ is even is the same as in the first solution.

5 points.

Notes on marking:

- Note that the set of solutions can also be characterized as the set of all pairs (n,k) such that $\nu_2(n) \leqslant \nu_2(k)$, where $\nu_2(x)$ denotes the largest nonnegative integer y such that $2^y \mid x$. Of course, this characterization or any other trivially equivalent characterization of the set of solutions is valid.

Problem 3. Let p be a prime number. Troy and Abed are playing a game. Troy writes a positive integer X on the board, and gives a sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers to Abed. Abed now makes a sequence of moves. The n-th move is the following:

Replace Y currently written on the board with either $Y + a_n$ or $Y \cdot a_n$.

Abed wins if at some point the number on the board is a multiple of p. Determine whether Abed can win, regardless of Troy's choices, if

a) $p = 10^9 + 7$;

b) $p = 10^9 + 9$.

Remark: Both $10^9 + 7$ and $10^9 + 9$ are prime.

(Ivan Novak)

Solution. We will prove that Abed cannot win in either case.

0 points.

We now explain Troy's strategies. Throughout the solution, we will use fractions modulo p.

a) Suppose $p = 10^9 + 7$. Note that $p \equiv 2 \pmod 3$. Let $X = 2$. We will define the sequence $(a_n)_{n\in\mathbb{N}}$ recursively. Note that neither 2 nor $2 - 1$ is divisible by p.

Suppose we've defined $a_1, \ldots, a_{n-1}$, where $n \in \mathbb{N}$, and suppose that whatever Abed's first $n-1$ moves are, the number on the board after these $n-1$ moves is congruent to Y modulo p, and neither Y nor $Y-1$ are divisible by p.
We now prove that there exists a positive integer k such that $Y + k \equiv Yk \pmod p$, and neither Yk nor $Yk - 1$ are not divisible by p.

Indeed, let $k \equiv \frac{Y}{Y-1} \pmod p$. Note that this is well defined since $Y - 1$ is not divisible by p. Then $Y + k \equiv Yk \equiv \frac{Y^2}{Y-1} \pmod p$. Note that $\frac{Y^2}{Y-1} \not\equiv 0 \pmod p$ since $Y \not\equiv 0 \pmod p$.

1 point.

Suppose for the sake of contradiction that $\frac{Y^2}{Y-1} \equiv 1 \pmod p$. This implies that $p \mid Y^2 - Y + 1$. However, this would imply $p \mid (-Y)^3 - 1$.
This means that $\mathrm{ord}_p(-Y) \mid 3$. Since $p \equiv 2 \pmod 3$ and $\mathrm{ord}_p(-Y) \mid p - 1$, it follows that $\mathrm{ord}_p(-Y) \neq 3$. This forces $\mathrm{ord}_p(-Y) = 1$. However, then $Y \equiv -1 \pmod p$, which implies $Y^2 - Y + 1 \equiv 3 \not\equiv 0 \pmod p$. Therefore, $\frac{Y^2}{Y-1} \not\equiv 1 \pmod p$.

2 points.

We define $a_n := k$. No matter what Abed's first n moves are, the number on the board after n moves is congruent to $\frac{Y^2}{Y-1}$ modulo p, which is not congruent to 0 or 1 modulo p. Therefore, Abed cannot win after n steps. Since this claim is true for any positive integer n, we conclude that Abed cannot win.

1 point.

b) Suppose $p = 10^9 + 9$. Note that $p \equiv 1 \pmod 4$, which means that there exists a positive integer z such that $z^2 \equiv -1 \pmod p$. Then there also exists a positive integer t such that $(2t-1)^2 \equiv -1 \pmod p$.
Let $X = t$. Note that neither X nor $X - 1$ are divisible by p, and note that $4X^2 - 4X + 2 \equiv 0 \pmod p$.

1 point.

Let $a_1 \equiv \frac{X}{X-1} \pmod p$. Then $a_1 + X \equiv a_1 X \equiv \frac{X^2}{X-1} \pmod p$. Therefore, whatever Abed's first move is, the number on the board after the first move will be congruent to $\frac{X^2}{X-1}$ modulo p. Furthermore, $\frac{X^2}{X-1}$ is not divisible by p since X isn't. Suppose for the sake of contradiction that $\frac{X^2}{X-1} \equiv 1 \pmod p$. Then $4X^2 - 4X + 4 \equiv 0 \pmod p$, but, by definition of X, $4X^2 - 4X + 2 \equiv 0 \pmod p$, which implies $2 \equiv 0 \pmod p$, which is a contradiction. Therefore, $\frac{X^2}{X-1} \not\equiv 1 \pmod p$.

1 point.

Let $a_2 \equiv \frac{X^2}{X^2-X+1} \pmod p$. Note that this is well defined since $X^2 - X + 1 \not\equiv 0 \pmod p$. Whatever Abed's second move is, the number on the board will be congruent to $\frac{X^2}{X-1} + \frac{X^2}{X^2-X+1} \equiv \frac{X^4}{(X^2-X+1)(X-1)} \pmod p$. Now note that

$$\frac{X^4}{(X^2 - X + 1)(X - 1)} \equiv X \pmod p \iff X^3 \equiv (X^2 - X + 1)(X - 1) \pmod p \iff 2X^2 - 2X + 1 \equiv 0 \pmod p,$$

which is true by definition of X. Therefore, whatever Abed's first two moves are, the number written on the board after the first two moves will be congruent to X modulo p.

4 points.

Thus, if we define $a_{2j-1} := a_1$ and $a_{2j} := a_2$ for $j \geqslant 2$, no matter what moves Abed makes, the number on the board will never be divisible by p.

0 points.

Notes on marking:

- Part a) is worth **4 points**, and part b) is worth **6 points**.
- The idea of making it impossible for Abed to affect the numbers on the board modulo p, although used in both parts, is worth **0 points** on its own.
- In part a), if a student doesn't prove that $x^2 - x + 1$ doesn't have prime divisors of the form $3k + 2$, but instead states that this fact is well known and checks that $10^9 + 7$ is of the form $3k + 2$, they should be awarded all the points intended for this part.
- In part b), the idea of 2-periodicity of the game state is worth **0 points** on its own.
- Due to overlapping arguments, if a student solves b), but does not solve a), then they get **0 points** for the very first point in part a). This point is then merged with the second block of **2 points** in part a).

Problem 4. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x+y) + f(xf(y)+1) = f(xf(x))$$

for all $x, y \in \mathbb{R}^+$.

(Amadej Kristjan Kocbek, Jakob Jurij Snoj)

First Solution. Let f be a function satisfying the equation. We split the solution into a series of claims.
Claim 1. $f(x) < f(f(1))$ for all $x > 1$.
Proof. Substituting $x = 1$ gives

$$f(y+1) + f(f(y)+1) = f(f(1)). \tag{3}$$

Since the function only attains positive values, we have $f(y+1) < f(f(1))$ for all y, and the conclusion follows. □

Claim 2. The function f is injective.
Proof. Assume the contrary and choose $a < b$ such that $f(a) = f(b)$. Substituting $y = a$ and, afterwards, $y = b$ into the original equations and comparing the equations gives

$$f(x+a) = f(x+b) \quad \text{for all } x \in \mathbb{R}^+.$$

Hence, f is periodic for all $x \geqslant P$ for some constant $P \in \mathbb{R}^+$ with period $p = b - a$. Fix some $x_1, y_1 \in \mathbb{R}^+$ with $x_1 > P$ and pick a positive integer n such that $(x_1 + np)f(x_1 + y_1) \geqslant f(f(1))$ and $(x_1 + np)f(x_1) > 1$. Substituting $x = x_1 + np, y = y_1$ gives

$$(x_1+np)f((x_1+np)+y_1) + f((x_1+np)f(y_1)+1) = f((x_1+np)f(x_1+np))$$
$$(x_1+np)f(x_1+y_1) + f((x_1+np)f(y_1)+1) = f((x_1+np)f(x_1))$$

after using periodicity to simplify the equation. Due to our choice of n and the function only attaining positive values, we have

$$f((x_1+np)f(x_1)) > (x_1+np)f(x_1+y_1) > f(f(1)).$$

However, since we have $(x_1+np)f(x_1) > 1$, Claim 1 implies $f((x_1+np)f(x_1)) < f(f(1))$, leading to a contradiction. Therefore, such a and b do not exist and f is injective. □

2 points.

Claim 3. $f(f(x)) = x$ for all $x \in \mathbb{R}^+$.
Proof. We substitute $y = f(y)$ into (1). Comparing the resulting equation with (1) gives:

$$f(f(y)+1) + f(f(f(y))+1) = f(y+1) + f(f(y)+1)$$
$$f(f(f(y))+1) = f(y+1)$$

Using injectivity, we get $f(f(y)) = y$ for all $y \in \mathbb{R}^+$. □

1 point.

Claim 4. For all $x \in \mathbb{R}^+$, $xf(x) \leqslant 1$. In particular, $f(a) \leqslant \frac{1}{x}$ for all $a \geqslant x$.
Proof. Assume the contrary - there exists some $c \in \mathbb{R}^+$ such that $cf(c) > 1$. Substituting $y = f(y)$ and using Claim 3, we transform the original equation:

$$xf(x+f(y)) + f(xy+1) = f(xf(x)).$$

Substituting $x = c, y = \frac{cf(c)-1}{c}$ into the above equation gives $cf(c + f((cf(c)-1)/c)) = 0$ after cancellation of the terms, a clear contradiction. The second part of the claim follows immediately. □

1 point.

Claim 5. For all $x \in \mathbb{R}^+$, we have $f(xf(x)) \leqslant 1$.
Proof. We notice

$$f(xf(x)) = xf(x+y) + f(xf(y)+1) < xf(x+y) + 1 \leqslant \frac{x}{x+y} + 1,$$

where the inequalities hold due to Claim 1 and Claim 4, respectively, as well as the identity $f(f(1)) = 1$. Assume there exists a c such that $f(cf(c)) > 1$: therefore, it should hold that

$$f(cf(c)) < \frac{c}{c+y} + 1.$$

However, the left hand side of the above inequality is independent of y. Thus, for y sufficiently large, the opposite direction of the inequality will hold since $c/(c+y)$ can get arbitrarily small, which leads to a contradiction. □

1 point.

Claim 6. For all $x \in \mathbb{R}^+$, $f(xf(x)) \geqslant 1$.

Proof. Assume the contrary. Therefore, there exists some a such that $f(af(a)) < 1$, let $f(af(a)) = 1 - e$. By Claim 4, there exists a $Y \in \mathbb{R}^+$ such that $f(y+1) < e$ for all $y > Y$. Let $d > Y$. Observing (1) after substituting $y = d$, we notice

$$f(f(d)+1) = 1 - f(d+1) > 1 - e.$$

Substituting $x = a, y = f\left(\frac{f(d)}{a}\right)$ into the original equation gives

$$1 - e = f(af(a)) = af\left(a + f\left(\frac{f(d)}{a}\right)\right) + f(f(d)+1) > 1 - e,$$

a contradiction. □

4 points.

Finally, observe Claims 5 and 6 together yield $f(xf(x)) = 1$ for all $x \in \mathbb{R}^+$. By injectivity, $xf(x)$ is constant, hence $f(x) = \frac{c}{x}$ for some constant $c \in \mathbb{R}^+$. By checking, we see $c = 1$ yields the only valid solution, $f(x) = \frac{1}{x}$.

1 point.

Second Solution. We present an alternative way of proving $f(xf(x))$ is constant after obtaining the first four claims of the first solution.

Assume there exist a and b such that $f(af(a)) - f(bf(b)) \neq 0$. Without loss of generality, we can assume $f(af(a)) - f(bf(b)) > 0$. We now substitute (x, y) with $\left(a, f\left(\frac{x}{a}\right)\right)$ and $\left(b, f\left(\frac{x}{b}\right)\right)$ and subtract the resulting equations to obtain

$$\begin{aligned}
f(af(a)) - f(bf(b)) &= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right) + f\left(af\left(f\left(\frac{x}{a}\right)\right) + 1\right) - f\left(bf\left(f\left(\frac{x}{b}\right)\right) + 1\right) \\
&= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right) + f\left(a \cdot \frac{x}{a} + 1\right) - f\left(b \cdot \frac{x}{b} + 1\right) \\
&= af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right).
\end{aligned}$$

1 point.

This shows that, as x varies, the expression $af\left(a + f\left(\frac{x}{a}\right)\right) - bf\left(b + f\left(\frac{x}{b}\right)\right)$ is constant. As f is an involution and thus surjective, we can choose a number $x_1 \in \mathbb{R}^+$ such that $a + f\left(\frac{x_1}{a}\right) > \frac{a}{f(af(a)) - f(bf(b))}$. Substituting x with x_1 in the above equation and using Claim 4, we obtain

$$\begin{aligned}
f(af(a)) - f(bf(b)) &= af\left(a + f\left(\frac{x_1}{a}\right)\right) - bf\left(b + f\left(\frac{x_1}{b}\right)\right) \\
&< af\left(a + f\left(\frac{x_1}{a}\right)\right) \\
&\leqslant \frac{a}{a + f\left(\frac{x_1}{a}\right)} \\
&< f(af(a)) - f(bf(b)),
\end{aligned}$$

which leads to a contradiction. Therefore, $f(xf(x))$ is constant.

4 points.

As in the first solution, this now implies $xf(x)$ is constant, therefore, f is of the form $f(x) = \frac{c}{x}$ for some constant c. We can easily check $f(x) = \frac{1}{x}$ is the only valid solution.

1 point.

Notes on marking:

- If a student doesn't check that $f(x) = \frac{1}{x}$ is indeed a solution or at least mention that it can be easily checked, they should lose **1 point**.
- Points from two marking schemes are not additive.