9TH EUROPEAN MATHEMATICAL CUP

12th December 2020 - 20th December 2020

Senior Category

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Problem 1. Let $ABCD$ be a parallelogram such that $|AB| > |BC|$. Let O be a point on the line CD such that $|OB| = |OD|$. Let ω be a circle with center O and radius $|OC|$. If T is the second intersection of ω and CD, prove that AT, BO and ω are concurrent.

(Ivan Novak)

Problem 2. Let n and k be positive integers. An n-tuple $(a_1, a_2, \ldots, a_n)$ is called a *permutation* if every number from the set $\{1, 2, \ldots, n\}$ occurs in it exactly once. For a permutation $(p_1, p_2, \ldots, p_n)$, we define its k-*mutation* to be the n-tuple

$$(p_1 + p_{1+k}, p_2 + p_{2+k}, \ldots, p_n + p_{n+k}),$$

where **indices** are taken modulo n. Find all pairs (n, k) such that every two distinct permutations have distinct k-mutations.

Remark: For example, when $(n, k) = (4, 2)$, the 2-mutation of $(1, 2, 4, 3)$ is $(1+4, 2+3, 4+1, 3+2) = (5, 5, 5, 5)$.

(Borna Šimić)

Problem 3. Let p be a prime number. Troy and Abed are playing a game. Troy writes a positive integer X on the board, and gives a sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers to Abed. Abed now makes a sequence of moves. The n-th move is the following:

Replace Y currently written on the board with either $Y + a_n$ or $Y \cdot a_n$.

Abed wins if at some point the number on the board is a multiple of p. Determine whether Abed can win, regardless of Troy's choices, if

a) $p = 10^9 + 7$;

b) $p = 10^9 + 9$.

Remark: Both $10^9 + 7$ and $10^9 + 9$ are prime.

(Ivan Novak)

Problem 4. Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x+y) + f(xf(y)+1) = f(xf(x))$$

for all $x, y \in \mathbb{R}^+$.

(Amadej Kristjan Kocbek, Jakob Jurij Snoj)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.