# 9TH European Mathematical Cup

*12th December 2020 - 20th December 2020*

Junior Category

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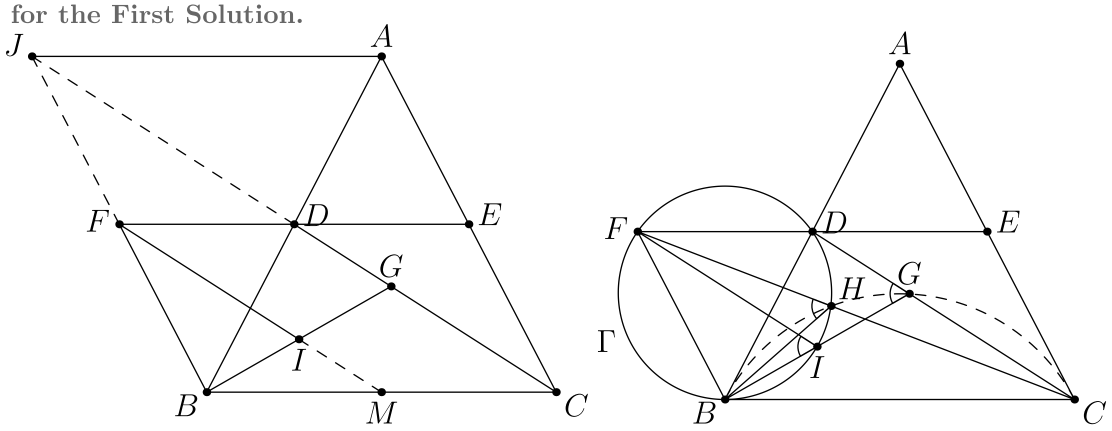
## Problems and Solutions

**Problem 1.** Let $ABC$ be an acute-angled triangle. Let $D$ and $E$ be the midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively. Let $F$ be the point such that $D$ is the midpoint of $\overline{EF}$. Let $\Gamma$ be the circumcircle of triangle $FDB$. Let $G$ be a point on the segment $\overline{CD}$ such that the midpoint of $\overline{BG}$ lies on $\Gamma$. Let $H$ be the second intersection of $\Gamma$ and $FC$. Show that the quadrilateral $BHGC$ is cyclic.

*(Art Waeterschoot, Belgium)*

**Sketch for the First Solution.**




**First Solution.** Since $D$ and $E$ are midpoints, the diagonals $\overline{AB}$ and $\overline{EF}$ of the quadrilateral $AFBE$ bisect each other, so $AFBE$ is a parallelogram. Hence $BF \parallel AE$.

**2 points.**

**Lemma.** If $I$ is the second intersection of $\Gamma$ and $\overline{BG}$, then $FI \parallel CD$. (We will present two different proofs.)

**First proof.** Let $J$ be the point such that $BCAJ$ is a parallelogram. Since $BF \parallel AE$, we have that $B$, $F$, $J$ are colinear.

**2 points.**

Since $D$ is the midpoint of $\overline{AB}$, $C$, $D$, $J$ are collinear.

**1 point.**

As $F$ and $I$ are midpoints of $\overline{BJ}$ and $\overline{BG}$, then $FI \parallel CD$. □

**2 points.**

**Second proof.** Let $M$ be the midpoint of $\overline{BC}$. As

$$|MC| = \frac{|BC|}{2} = |DE| = |DF|$$

and $FD \parallel MC$, then $MCDF$ is a parallelog., so $MF \parallel CD$.

**2 points.**

As $M$ and $I$ are midpoints of $\overline{BC}$ and $\overline{BG}$, then $MI \parallel CD$.

**2 points.**

Hence $M$, $I$ and $F$ are collinear and $FI \parallel CD$. □

**1 point.**

Now as we know that $FI \parallel CD$, we have $\angle BIF = \angle BGD$.

**1 point.**

As $BIHF$ is a cyclic quadrilateral, we have $\angle BIF = \angle BHF$.

**1 point.**

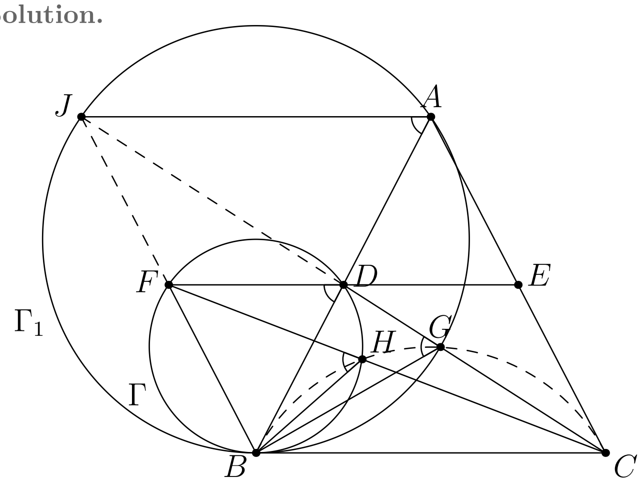
Hence

$$\angle CHB = 180^\circ - \angle BHF = 180^\circ - \angle BGD = \angle CGB,$$

so $BHGC$ is cyclic as desired.

**1 point.**

**Sketch for the Second Solution.**




**Second Solution.** Since $D$ and $E$ are midpoints, the diagonals $\overline{AB}$ and $\overline{EF}$ of the quadrilateral $AFBE$ bisect each other, so $AFBE$ is a parallelogram. Hence $BF \parallel AE$.

**2 points.**

Let $J$ be the point such that $BCAJ$ is a parallelogram. Since $BF \parallel AE$, we have that $B$, $F$, $J$ are collinear.

**2 points.**

Since $D$ is the midpoint of $\overline{AB}$, $C$, $D$, $J$ are collinear.

**1 point.**

Now let $\Gamma_1$ be the circumcircle of triangle $JAB$. As $F$ and $D$ are midpoints of $\overline{BJ}$ and $\overline{BA}$, and the midpoint of $\overline{BG}$ lies on $\Gamma$, we can redefine $G$ as the second intersection of $\Gamma_1$ and $CJ$.

**2 points.**

As $AJBG$ is a cyclic quadrilateral, we have $\angle BGJ = \angle BAJ$.

**1 point.**

As $FD$ is parallel to $JA$, we have $\angle BAJ = \angle BDF$.

**0 points.**

As $BHDF$ is a cyclic quadrilateral, we have $\angle BDF = \angle BHF$.

**1 point.**

Hence

$$\angle CHB = 180^\circ - \angle BHF = 180^\circ - \angle BGD = \angle CGB,$$

so $BHGC$ is cyclic as desired.

**1 point.**

**Notes on marking:**

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a complete solution can be awarded.

**Problem 2.** A positive integer $k \geqslant 3$ is called *fibby* if there exists a positive integer $n$ and positive integers $d_1 < d_2 < \ldots < d_k$ with the following properties:

- $d_{j+2} = d_{j+1} + d_j$ for every $j$ satisfying $1 \leqslant j \leqslant k-2$,
- $d_1, d_2, \ldots, d_k$ are divisors of $n$,
- any other divisor of $n$ is either less than $d_1$ or greater than $d_k$.

Find all fibby numbers.

*(Ivan Novak)*

**Solution.** Note that $(1, 2, 3, 5)$ is a sequence of length 4 such that all its elements are divisors of 30 and every other divisor of 30 is either less than 1 or greater than 5. Also $3 = 1 + 2$ and $5 = 2 + 3$, which means 4 is fibby. Consequently, 3 is also fibby.

**1 point.**

Suppose there exist positive integers $n$, $d_1 < d_2 < \ldots < d_k$ satisfying the problem's conditions, with $k \geqslant 5$. Suppose for the sake of contradiction that $d_j$ is even for some $j \geqslant 3$. Then $\frac{d_j}{2}$ is also a divisor of $n$.

**1 point.**

However,

$$d_1 \leqslant d_{j-2} < \frac{d_{j-1} + d_{j-2}}{2} = \frac{d_j}{2} < d_{j-1} < d_k.$$

This implies $\frac{d_j}{2}$ is a divisor of $n$ which is neither less than $d_1$ nor greater than $d_k$ and is distinct from the numbers $d_1, d_2, \ldots, d_k$, which is a contradiction.

**6 points.**

This implies that $d_3$ and $d_4$ are odd. However, this means that $d_5 = d_3 + d_4$ is even, which is a contradiction. Therefore, any number greater than 4 is not fibby.

**2 points.**

**Notes on marking:**

- The part of the proof where we prove all $k \geq 5$ are not fibby is worth **9 points**. It may happen that a contestant proves a weaker statement in that direction.
  - If a contestant proves that there exists $C$ such that no $k \geq C$ is fibby, they should get **1 point**.
  - If the $C$ above is explicit, they should get an additional **1 point**.
  - If in addition $C = 6$, they should get **1 point** more.

  The points above (at most **3 points**) are not additive with the points for proving $C = 5$ in the official solution. Thus, without using ideas that can solve the $C = 5$ case, the contestant should not get more than **1 point** for the construction, plus the points above if applicable.
- Many solutions proceed by cases on the parity of $d_1$ and $d_2$. However, in all solutions that the Problem Selection Committee were aware of, the only parity that matters is the parity of some $d_j, j \geq 3$.

  Thus, stating and proving that some of $d_3, d_4$ and $d_5$ is even is worth **2 points**, as in the official solution, and no other points are awarded for parity concerns.

**Problem 3.** Two types of tiles, depicted on the figure below, are given.

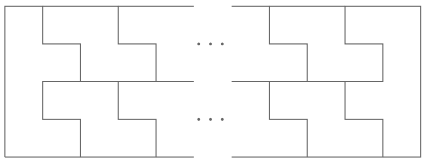

Find all positive integers $n$ such that an $n \times n$ board consisting of $n^2$ unit squares can be covered without gaps with these two types of tiles (rotations and reflections are allowed) so that no two tiles overlap and no part of any tile covers an area outside the $n \times n$ board.

*(Art Waeterschoot)*

**Solution.** We claim such a tiling exists whenever $n$ is divisible by 4 and greater than 4.

**0 points.**

We now prove the existence of a tiling in the case where $n$ is divisible by 4 and greater than 4. The figure below shows that if $k \geq 1$, we can tile a $(2k+1) \times 4$-rectangle.



**1 point.**

By gluing a $3 \times 4$ rectangle to the above tiling, we get a tiling of any $(4k+4) \times 4$ rectangle, where $k \geqslant 1$. We can now stack $k+1$ such rectangles next to each other to obtain a $(4k+4) \times (4k+4)$ square, which proves the claim.

**1 point.**

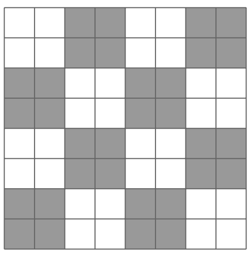
Suppose we can tile a $n \times n$ square with the given tiles. Let $a$ and $b$ be the number of $F$-tiles and $Z$-tiles used in the tiling, respectively. Then $6a + 4b = n^2$, which implies $n$ is even. This implies that $a$ is also even. Let $n = 2k$, where $k$ is a positive integer.

**0 points.**

Consider the following colouring of the square: divide up the square into $k^2$ smaller squares of size $2 \times 2$ and colour these squares with a chessboard colouring (see the figure below). Every $F$-tile covers exactly 3 black unit squares and every $Z$-tile covers an odd number of black unit squares.

**1 point.**

Because there are an even number of black squares, we obtain that $a$ and $b$ have equal parity. Since $a$ is even, this implies that $b$ is even.

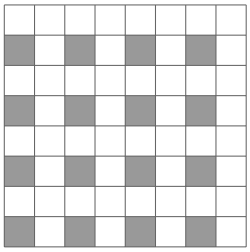


**3 points.**

Now colour all unit squares in an even row and odd column black (see the figure below). Now every $F$-tile covers an even number of black unit squares and every $Z$-tile covers exactly one black unit square.

**1 point.**

Since the number of black squares is $k^2$, we obtain that $b$ and $k^2$ have equal parity. Since $b$ is even, this implies $k$ is even.



**3 points.**

Therefore, $n$ is a multiple of 4.

**0 points.**

Furthermore, it is easily seen a $4 \times 4$-square cannot be tiled, as there are no positive integers $(a, b)$ such that $b$ is even and $6a + 4b = 16$.

**0 points.**

**Notes on marking:**

- Colouring a square in a certain way without drawing any relevant conclusions from the colouring is worth **0 points**.
- Another possible solution is to consider a colouring with 4 colours by dividing up into small $2 \times 2$-squares. In fact this is equivalent to our solution, because is the same as considering both colourings above at once. Considering such a colouring and drawing the same conclusions is worth the same amount of points as considering the colourings one by one.
- If a student doesn't check the case when $n = 4$, they can score at most **9 points** on the problem.
- The standard chessboard colouring gives only that $a$ is even, which is considered trivial by the Jury, thus it is worth **0 points**.
- If a student has another colouring which proves that $2|b$, this is worth **4 points**, as in the official solution.
- If a student has another colouring which proves that $4|a$, this is worth **4 points**, as in the official solution.

**Problem 4.** Let $a, b, c$ be positive real numbers such that $ab+bc+ac = a+b+c$. Prove the following inequality:

$$\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \leqslant \sqrt{2}\cdot \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}.$$

*(Dorlir Ahmeti)*

**First Solution.** We can rewrite the inequality as

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} \leqslant 4\cdot \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$$

and distinguish two cases based on what the right hand side is.

**Case 1.** $\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} = \frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.
Using *AM-GM* inequality, we have

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} \leqslant \sum_{cyc}\left(2+a+\frac{b}{c}\right) = 6+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

**2 points.**

Hence, it is enough to prove

$$6+a+b+c+\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \leqslant 4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$$
$$\iff 6+a+b+c \leqslant 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right). \tag{1}$$

Applying *AM-GM* inequality we obtain

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant 2\cdot 3\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}} = 6 \tag{2}$$

and using *Cauchy-Schwarz* inequality together with the condition allows us to conclude:

$$(ab+bc+ac)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant (a+b+c)^2 = (a+b+c)(ab+bc+ac)$$
$$\implies \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant a+b+c. \tag{3}$$

**2 points.**

Combining results (2) and (3) yields (1).

**Case 2.** $\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\} = \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$.
Using *AM-GM* inequality, we have

$$\sum_{cyc} 2\sqrt{2\left(a+\frac{b}{c}\right)} = \sum_{cyc} 2\sqrt{\frac{2a}{c}\left(c+\frac{b}{a}\right)} \leqslant \sum_{cyc}\left(\frac{2a}{c}+c+\frac{b}{a}\right) = a+b+c+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

**4 points.**

Hence, it is enough to prove

$$a+b+c+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \leqslant 4\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$
$$\iff a+b+c \leqslant \frac{b}{a}+\frac{c}{b}+\frac{a}{c}.$$

Using *Cauchy-Schwarz* inequality together with the condition allows us to conclude

$$(ab+bc+ac)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \geqslant (a+b+c)^2 = (a+b+c)(ab+bc+ac)$$
$$\implies \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \geqslant a+b+c$$

**2 points.**

which is exactly what we wanted to prove.

**Second Solution.** Using the substitution $m = \min\left\{\frac{a}{b} + \frac{b}{c} + \frac{c}{a},\ \frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right\}$, we can rewrite the inequality as

$$\frac{1}{3}\left(\sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}}\right) \leqslant \frac{m\sqrt{2}}{3}.$$

Recognizing the left hand side as an arithmetic mean, we may apply the *QM-AM* inequality to obtain

$$\frac{1}{3}\left(\sqrt{a + \frac{b}{c}} + \sqrt{b + \frac{c}{a}} + \sqrt{c + \frac{a}{b}}\right) \leqslant \sqrt{\frac{a + \frac{b}{c} + b + \frac{c}{a} + c + \frac{a}{b}}{3}}.$$

We're now left with proving

$$\frac{a + \frac{b}{c} + b + \frac{c}{a} + c + \frac{a}{b}}{3} \leqslant \left(\frac{m\sqrt{2}}{3}\right)^2$$

which can be written as:

$$\frac{3}{2}(a + b + c) + \frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leqslant m^2. \tag{1}$$

**1 point.**

We distinguish two cases based on the value of $m$:

**Case 1.** $m = \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$.

Expanding the right hand side of (1) and cancelling out $\frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)$ turns the inequality into

$$\frac{3}{2}(a + b + c) \leqslant \frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} + \frac{1}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Multiplying both sides by $2(ab + bc + ac)$ and making use of the given condition on the left hand side gives us:

$$3(a + b + c)^2 \leqslant 2\left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)(ab + bc + ac) + \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)(ab + bc + ac).$$

We may now apply *Cauchy-Schwarz* inequality to obtain $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)(ab + bc + ac) \geqslant (a + b + c)^2$

**2 points.**

and this leaves us with proving the following:

$$(a + b + c)^2 \leqslant \left(\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2}\right)(ab + bc + ac). \tag{2}$$

We now make use of a well known lemma:

**Lemma 1.** *For positive real numbers $x, y, z$ one has* $\frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geqslant \frac{x + y + z}{\sqrt[3]{xyz}}$.

*Proof.* Applying *AM-GM* inequality we obtain:

$$\frac{x}{y} + \frac{x}{y} + \frac{y}{z} \geqslant 3\sqrt[3]{\frac{x^2 y}{y^2 z}} = \frac{3x}{\sqrt[3]{xyz}},$$
$$\frac{y}{z} + \frac{y}{z} + \frac{z}{x} \geqslant 3\sqrt[3]{\frac{y^2 z}{z^2 x}} = \frac{3y}{\sqrt[3]{xyz}},$$
$$\frac{z}{x} + \frac{z}{x} + \frac{x}{y} \geqslant 3\sqrt[3]{\frac{z^2 x}{x^2 y}} = \frac{3z}{\sqrt[3]{xyz}}.$$

Summing up the above three inequalities finishes the proof of the lemma. □

**3 points.**

Applying the lemma we obtain $\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} \geqslant \frac{a^2 + b^2 + c^2}{\sqrt[3]{a^2 b^2 c^2}}$ and applying *AM-GM* we obtain $ab + bc + ac \geqslant 3\sqrt[3]{a^2 b^2 c^2}$, which together used in (2) mean that we only need to prove

$$(a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2)$$

and this is equivalent to $(a - b)^2 + (b - c)^2 + (a - c)^2 \geqslant 0$.

**1 point.**

**Case 2.** $m = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.

Expanding the right hand side of (1) turns the inequality into

$$\frac{3}{2}(a+b+c) + \frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leqslant \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right). \quad (3)$$

Since $m = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, we have that $\frac{3}{2}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leqslant \frac{3}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)$ and using this in (3), we're left with proving:

$$\frac{3}{2}(a+b+c) \leqslant \frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{1}{2}\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right).$$

**1 point.**

The rest of the proof is now analogous to the steps we used to solve the first case, namely multiplying both sides by $2(ab+bc+ac)$ and making use of the given condition, applying *Cauchy-Schwarz* inequality to prove $\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)(ab+bc+ac) \geqslant (a+b+c)^2$, making use of the lemma to prove $\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geqslant \frac{a^2+b^2+c^2}{\sqrt[3]{a^2b^2c^2}}$, making use of *AM-GM* inequality to obtain $ab+bc+ac \geqslant 3\sqrt[3]{a^2b^2c^2}$ and finally proving $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$.

**2 points.**

**Third Solution.** Let $m = \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$ and $n = \max\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}$.

**0 points.**

Using *Cauchy-Schwarz* inequality, we obtain the following:

$$\begin{aligned}\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} &= \sqrt{\left(\sqrt{\frac{ac+b}{c}}+\sqrt{\frac{ab+c}{a}}+\sqrt{\frac{bc+a}{b}}\right)^2}\\ &\leqslant \sqrt{(ac+b+ab+c+bc+a)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)}\end{aligned}$$

**2 points.**

Now by using $ab+bc+ac=a+b+c$, we get:

$$\sqrt{(ac+b+ab+c+bc+a)\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)} = \sqrt{2\,(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}.$$

Therefore, we want to show

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leqslant m. \tag{1}$$

**0 points.**

We proceed by proving

$$m^2 \geqslant 3+2n. \tag{2}$$

*Proof.* Using *AM-GM* inequality, we get the following:

$$\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2} \geqslant 3.$$

Applying this result, we see that

$$\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^2 = \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geqslant 3+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Analogously, we also get that $\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geqslant 3+2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$, which proves (2). □

**2 points.**

Now $m \leqslant n$ along with (2) yields

$$\begin{aligned}\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} &= \sqrt{3+\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\\ &= \sqrt{3+m+n}\\ &\leqslant \sqrt{3+2n}\\ &\leqslant \sqrt{m^2} = m\end{aligned}$$

which is exactly (1).

**6 points.**

**Notes on marking:**

- In the third solution, considering only one case for $m \neq n$ and completing the proof is worth **8 points**. Full points are awarded if the analogy to the other case is mentioned.
- Proving $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant 3$ should not be awarded any points as this claim is considered trivial.
- In the first solution, proving $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant a+b+c$ (or the analogous version) and not applying this inequality in both cases such that the application leads to the solution should only be awarded **2 points**.