9^{TH} European Mathematical Cup

12^{th} December 2020 - 20^{th} December 2020

Junior Category

Problem 1. Let ABC be an acute-angled triangle. Let D and E be the midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively. Let F be the point such that D is the midpoint of $\overline{EF}$. Let Γ be the circumcircle of triangle FDB. Let G be a point on the segment $\overline{CD}$ such that the midpoint of $\overline{BG}$ lies on Γ. Let H be the second intersection of Γ and FC. Show that the quadrilateral $BHGC$ is cyclic.

(Art Waeterschoot)

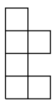
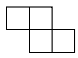
Problem 2. A positive integer $k \geqslant 3$ is called *fibby* if there exists a positive integer n and positive integers $d_1 < d_2 < \ldots < d_k$ with the following properties:

- $d_{j+2} = d_{j+1} + d_j$ for every j satisfying $1 \leqslant j \leqslant k-2$,
- $d_1, d_2, \ldots, d_k$ are divisors of n,
- any other divisor of n is either less than d_1 or greater than d_k.

Find all fibby numbers.

(Ivan Novak)

Problem 3. Two types of tiles, depicted on the figure below, are given.

Tile F:  Tile Z: 

Find all positive integers n such that an $n \times n$ board consisting of n^2 unit squares can be covered without gaps with these two types of tiles (rotations and reflections are allowed) so that no two tiles overlap and no part of any tile covers an area outside the $n \times n$ board.

(Art Waeterschoot)

Problem 4. Let a, b, c be positive real numbers such that $ab+bc+ac = a+b+c$. Prove the following inequality:

$$\sqrt{a+\frac{b}{c}} + \sqrt{b+\frac{c}{a}} + \sqrt{c+\frac{a}{b}} \leqslant \sqrt{2} \cdot \min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}.$$

(Dorlir Ahmeti)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.