8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Senior Category

MLADI NADARENI MATEMATIČARI
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Problem 1. For positive integers a and b, let $\gcd(a, b)$ denote their greatest common divisor. Determine all pairs of positive integers (m, n) such that for any two positive integers x and y such that $x \mid m$ and $y \mid n$,

$$\gcd(x + y, mn) > 1.$$

(Ivan Novak)

Problem 2. Let n be a positive integer. A $n \times n$ board consisting of n^2 cells, each being a unit square coloured either black or white, is called *convex* if for every black coloured cell, both the cell directly to the left of it (if it exists) and the cell directly above it (if it exists) are also coloured black. We define the *beauty* of a board as the number of pairs of its cells (u, v) such that u is black, v is white and u and v are in the same row or column. Determine the maximum possible beauty of a convex $n \times n$ board.

(Ivan Novak)

Problem 3. In an acute triangle ABC with $|AB| \neq |AC|$, let I be the incenter and O the circumcenter. The incircle is tangent to $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ in D, E and F respectively. Prove that if the line parallel to EF passing through I, the line parallel to AO passing through D and the altitude from A are concurrent, then the point of concurrence is the orthocenter of the triangle ABC.

(Petar Nizić-Nikolac)

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f(yf(x) + f(y)) = f(x + 2f(y)) + xy$$

for all $x, y \in \mathbb{R}$.

(Adrian Beker)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.