8^{TH} European Mathematical Cup

14^{th} December 2019 - 22^{th} December 2019

Junior Category

MLADI NADARENI MATEMATIČARI
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Problem 1. Every positive integer is marked with a number from the set $\{0, 1, 2\}$, according to the following rule:

if a positive integer k is marked with j, then the integer $k + j$ is marked with 0.

Let S denote the sum of marks of the first 2019 positive integers. Determine the maximum possible value of S.

(Ivan Novak)

Problem 2. Define a sequence $x_1, x_2, x_3, \ldots$ such that $x_1 = \sqrt{2}$ and

$$x_{n+1} = x_n + \frac{1}{x_n} \text{ for } n \in \mathbb{N}.$$

Prove that the following inequality holds:

$$\frac{x_1^2}{2x_1x_2 - 1} + \frac{x_2^2}{2x_2x_3 - 1} + \ldots + \frac{x_{2018}^2}{2x_{2018}x_{2019} - 1} + \frac{x_{2019}^2}{2x_{2019}x_{2020} - 1} > \frac{2019^2}{x_{2019}^2 + \frac{1}{x_{2019}^2}}.$$

(Ivan Novak)

Problem 3. Let ABC be a triangle with circumcircle ω. Let l_B and l_C be two lines through the points B and C, respectively, such that $l_B \parallel l_C$. The second intersections of l_B and l_C with ω are D and E, respectively. Assume that D and E are on the same side of BC as A. Let DA intersect l_C at F and let EA intersect l_B at G. If O, O_1 and O_2 are circumcenters of the triangles ABC, ADG and AEF, respectively, and P is the circumcenter of the triangle OO_1O_2, prove that $l_B \parallel OP \parallel l_C$.

(Stefan Lozanovski)

Problem 4. Let u be a positive rational number and m be a positive integer. Define a sequence $q_1, q_2, q_3, \ldots$ such that $q_1 = u$ and for $n \geqslant 2$:

$$\text{if } q_{n-1} = \frac{a}{b} \text{ for some relatively prime positive integers } a \text{ and } b, \text{ then } q_n = \frac{a + mb}{b + 1}.$$

Determine all positive integers m such that the sequence $q_1, q_2, q_3, \ldots$ is eventually periodic for any positive rational number u.

Remark: A sequence $x_1, x_2, x_3, \ldots$ is *eventually periodic* if there are positive integers c and t such that $x_n = x_{n+t}$ for all $n \geqslant c$.

(Petar Nizić-Nikolac)

Time: 240 minutes.

Each problem is worth 10 points.

The use of calculators or any other instruments except rulers and compasses is not permitted.