7^{TH} European Mathematical Cup

8^{th} December 2018 - 16^{th} December 2018

Senior Category

Problems and Solutions

Problem 1. A partition of a positive integer is *even* if all its elements are even numbers. Similarly, a partition is *odd* if all its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of n.

Remark: A *partition* of a positive integer n is a non-decreasing sequence of positive integers whose sum of elements equals n. For example, $(2,3,4)$, $(1,2,2,2,2)$ and (9) are partitions of 9.

(Ivan Novak)

Solution. Answer: $n \in \{2,4\}$.

We first notice that if n is a solution, n must be even, otherwise there are no even partitions of n, and (n) is an odd partition, so the number of odd partitions is greater then the number of even partitions.

1 point.

We now construct an injection f from the set of even partitions of n of cardinality k to odd partitions of n of cardinality $2k$.

If $p = (a_1, \dots, a_k)$, where $2 \leqslant a_1 \leqslant a_2 \leqslant \dots \leqslant a_k$ is an even partition, let

$$f(p) = (\underbrace{1, \dots, 1}_{k \text{ times}}, a_1 - 1, \dots, a_k - 1).$$

5 points.

Obviously, $f(p)$ is an odd partition of n. It is easy to see that f is injective because if $f(p) = f(q)$ then the largest k elements of $f(p)$ and $f(q)$ are equal, and then p and q must be equal.

2 points.

Number of odd partitions is equal to the number of even partitions if and only if f is surjective.

1 point.

It can be checked that for $n = 2$, $n = 4$, f is a bijection. Check (no points deducted if missing):

$\underline{n = 2}$

$(2) \to (1,1)$

$\underline{n = 4}$

$(4) \to (1,3)$

$(2,2) \to (1,1,1,1)$

For $n > 4$, partition $(3, n-3)$ is not in the image of f, since every element of the image contains at least one number 1, so the number of even partition is equal to the the number of odd partitions if and only if $n \in \{2,4\}$.

1 point.

Notes on marking:

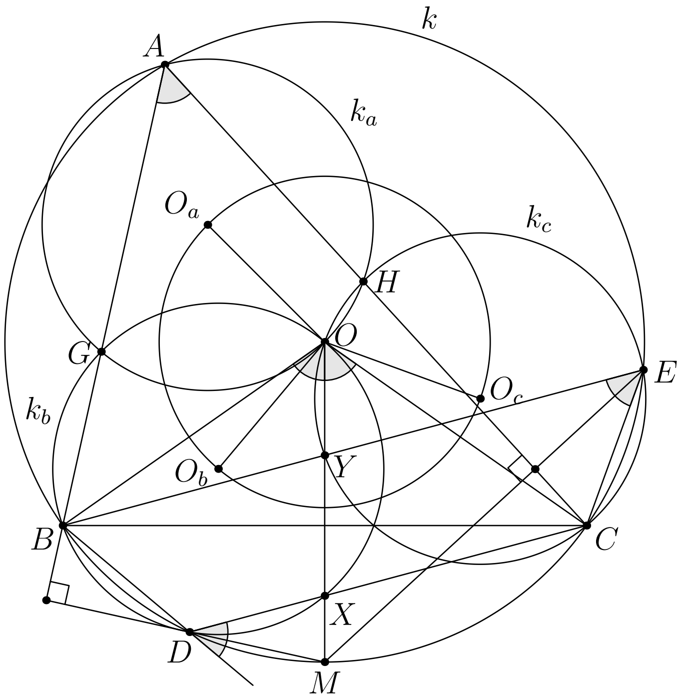
- Stating that $n = 2, 4$ are the only solutions on its own is worth **0 points**.
- Clearly attempting to construct an injection from the set of even partitions to the set of odd partitions without success is worth **1 point**.

Problem 2. Let ABC be a triangle with $|AB| < |AC|$. Let k be the circumcircle of $\triangle ABC$ and let O be the center of k. Point M is the midpoint of the arc $\widehat{BC}$ of k not containing A. Let D be the second intersection of the perpendicular line from M to AB with k and E be the second intersection of the perpendicular line from M to AC with k. Points X and Y are the intersections of CD and BE with OM respectively. Denote by k_b and k_c circumcircles of triangles BDX and CEY respectively. Let G and H be the second intersections of k_b and k_c with AB and AC respectively. Denote by k_a the circumcircle of triangle AGH.

Prove that O is the circumcenter of $\triangle O_aO_bO_c$, where O_a, O_b, O_c are the centers of k_a, k_b, k_c respectively.

(Petar Nizić-Nikolac)

First Sketch.



First Solution. We introduce standard angle notation, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$.

As M is midpoint of arc $\widehat{BC}$, we know that $\angle MOB = \angle COM = \frac{\angle COB}{2} = \angle CAB = \alpha$, so

$$180^\circ - \angle BDX = 180^\circ - \angle BDC = \angle BAC = \angle BOM = \angle BOX$$

implying that $BDXO$ is a cyclic quadrilateral. Analogously we get that $CEOY$ is a cyclic quadrilateral.

2 points.

Another property of M being a midpoint of arc $\widehat{BC}$ is that $\angle CAM = \angle MAB = \frac{\alpha}{2}$, so

$$\angle DAB = 180^\circ - \angle ABD - \angle BDA = (\angle BDM - 90^\circ) - \angle BCA = (90^\circ - \angle MAB) - \gamma = \left(90^\circ - \frac{\alpha}{2}\right) - \gamma = \frac{\beta - \gamma}{2} \quad (1)$$

$$\angle EAC = 180^\circ - \angle CEA - \angle ACE = \angle ABC - (90^\circ - \angle CEM) = \beta - (90^\circ - \angle CAM) = \beta - \left(90^\circ - \frac{\alpha}{2}\right) = \frac{\beta - \gamma}{2} \quad (2)$$

Combining (1) and (2) we obtain that $|BD| = |EC|$.

2 points.

As B,C,D and E lie on circumcircle, $|BO| = |CO| = |DO| = |EO|$, thus $\triangle BOD \cong \triangle COD$. As k_b and k_c are circumcircles of triangles BOD and COE respectively, we conclude that $k_b \cong k_c$, thus $|OO_b| = |OO_c|$.

2 points.

Now see that

$$\angle AGO = \angle ODB = 90^\circ - \frac{\angle DOB}{2} = 90^\circ - \angle DAB \quad (3)$$

$$\angle OHA = 180^\circ - \angle OEC = 180^\circ - \left(90^\circ - \frac{\angle EOC}{2}\right) = 90^\circ + \angle EAC \quad (4)$$

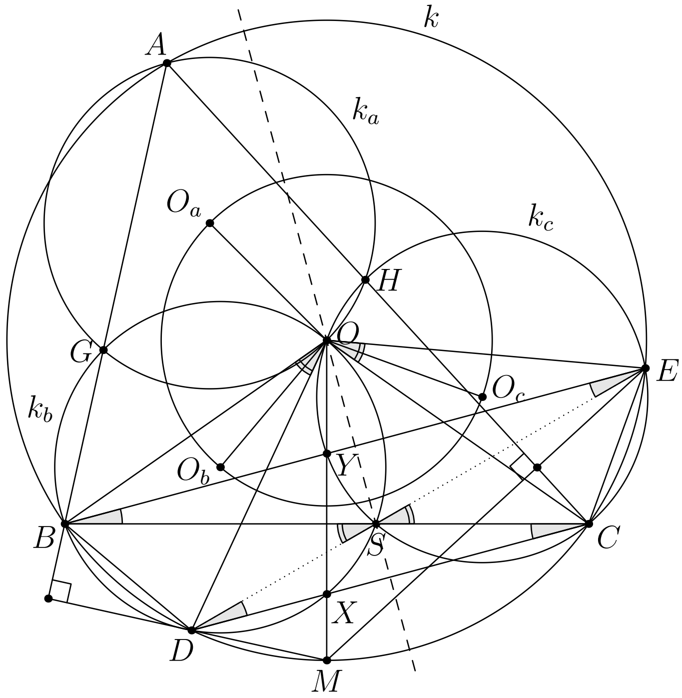
Combining (1), (2), (3) and (4) we obtain that $AGOH$ is a cyclic quadrilateral.

2 points.

Now as $|AO| = |BO|$ and $\angle AGO = \angle BDO$ we conclude that $k_a \cong k_b$, thus $|OO_a| = |OO_b| = |OO_c|$, so O is the circumcenter of $\triangle O_aO_bO_c$.

2 points.

Second Sketch.



Second Solution. We introduce standard angle notation, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. As M is midpoint of arc $\widehat{BC}$, we know that $\angle CAM = \angle MAB = \frac{\alpha}{2}$, so

$$\angle DAB = 180^\circ - \angle ABD - \angle BDA = (\angle BDM - 90^\circ) - \angle BCA = (90^\circ - \angle MAB) - \gamma = \left(90^\circ - \frac{\alpha}{2}\right) - \gamma = \frac{\beta - \gamma}{2} \quad (1)$$

$$\angle EAC = 180^\circ - \angle CEA - \angle ACE = \angle ABC - (90^\circ - \angle CEM) = \beta - (90^\circ - \angle CAM) = \beta - \left(90^\circ - \frac{\alpha}{2}\right) = \frac{\beta - \gamma}{2} \quad (2)$$

Combining (1) and (2) we obtain that $|BD| = |EC|$, so $BDCE$ is an isoscales trapezoid.

2 points.

Let S be the intersection of diagonals of $BDCE$. Then using (1) and (2) we have

$$\angle DSB = \angle SBE + \angle SDC = 2\angle EAC = 2\angle DAB = \angle DO_bD$$

so S lies on k_b. Analogously we get that S lies on k_c as well.

2 points.

Let O' be the second intersection of k_b and k_c. Then

$$\angle EO'B = \angle EO'S + \angle SO'B = 360^\circ - \angle SCE - \angle BDS = 2(180^\circ - \angle SCE) = 2(\angle EAB) = \angle EOB$$

and as k_b is symmetric to k_c over OS (perpendicular bisector of $\overline{BE}$ and $\overline{CE}$), we conclude that O and O' lie on that line so $O \equiv O'$, and we conclude that O is the second intersection of k_b and k_c.

2 points.

As k_b is symmetric to k_c over OS, we conclude that $|OO_b| = |OO_c|$.

1 point.

As $k_a = (AGH)$, $k_b = (BSOG)$ and $k_c = (CEOS)$, due to Miquel's theorem we have that O lies on k_a.

1 point.

Now as $|AO| = |BO|$ and $\angle AGO = \angle BDO$ we conclude that $k_a \cong k_b$, thus $|OO_a| = |OO_b| = |OO_c|$, so O is the circumcenter of $\triangle O_aO_bO_c$.

2 points.

Notes on marking:

- If a student has a partial solution with analytic methods, only points for proving facts that can be expressed in geometric ways and lead to a compete solution can be awarded.

Problem 3. For which real numbers $k > 1$ does there exist a bounded set of positive real numbers S with at least 3 elements such that

$$k(a-b) \in S$$

for all $a, b \in S$ with $a > b$?

Remark: A set of positive real numbers S is *bounded* if there exists a positive real number M such that $x < M$ for all $x \in S$.

(Petar Nizić-Nikolac)

First Solution. Set of solutions:

$$k \in \left\{\frac{1+\sqrt{5}}{2}, 2\right\}$$

Verification:

- If $k = \phi = \frac{1+\sqrt{5}}{2}$ we can choose set $\{\phi, 1+\phi, 1+2\phi\}$. It works as $\phi(1+\phi-\phi) = \phi$, $\phi(1+2\phi-1-\phi) = \phi^2 = 1+\phi$ and $\phi(1+2\phi-\phi) = \phi+\phi^2 = 1+2\phi$ (all these properties are true as ϕ is a root of the quadratic $x^2-x-1=0$).
- If $k = 2$ we can choose set $\{2, 3, 4\}$. It works as $2(3-2) = 2$, $2(4-3) = 2$ and $2(4-2) = 4$.

1 point.

Now we prove that these are the only possible values of k. Suppose $k > 1$ such that all required properties are satisfied.

Lemma 1. $k(a-b) \leqslant a$ for all $a, b \in S$ with $a > b$

Proof. Assume the opposite, that there exist $a, b \in S$ with $a > b$ such that $k(a-b) > a$. Fix b and denote $f(x) = k(x-b)$. We have $f(a) > a$. Consider these two conclusion for some x such that $f(x) > x$:

$$f(x) > x \implies k(x-b) - b > x - b \implies k(k(x-b)-b) - kb > k(x-b) - kb \implies f(f(x)) > f(x) \tag{1}$$

1 point.

$$f(x) > x \implies (k-1)f(x) > (k-1)x \implies k(f(x)-b) - k(x-b) > f(x) - x \implies f(f(x)) - f(x) > f(x) - x \tag{2}$$

1 point.

By (1) we have that $f^n(a) > f^{n-1}(a) > \ldots > f(a) > a > b$ so $f^n(a) \in S, \forall n \in \mathbb{N}$. On the other hand, by (2) we have $f^k(a) - f^{k-1}(a) \geqslant f(a) - a$ for all natural k. Summing up for k from 1 to n, we obtain

$$f^n(a) - a = \sum_{k=1}^{n} (f^k(a) - f^{k-1}(a)) \geqslant n(f(a) - a)$$

However, this means that $f^n(a) \in S$ is unbounded as n grows, which is impossible. Hence, the lemma is proved. □

1 point.

Lemma 2. S has a minimum and it is greater than 0

Proof. Now, denote $m = \inf S$. Let's first settle the case $m = 0$. However, then by fixing a and taking b small enough such that $k(a-b) > a$ we contradict the lemma. Therefore, we have $m > 0$.

1 point.

Without loss of generality we can take that $m = 1$ as we can scale the whole set. Assume that $1 \notin S$, and then there exists an infinite sequence of elements of S tending to 1, i.e., for every $a \in S$ there exists $b \in S$ with $1 < b < a$. Therefore,

$$k(a-b) > 1 \implies a > b + \frac{1}{k} \implies a > 1 + \frac{1}{k}$$

However, then every a in S is larger than $1 + \frac{1}{k}$ so $1 = \inf S \geqslant 1 + \frac{1}{k} > 1$, which is a contradiction. Hence $\min S = 1$. □

1 point.

Lemma 3. For some $x \in S$, if $x > G_{n-1}$ then $x \geqslant G_n$ for all $n \in \mathbb{N}$, where $G_n = 1 + \frac{1}{k} + \ldots + \frac{1}{k^n}$

Proof. We prove by induction on n. Basis for $n = 0$ is true as

$$k(x-1) \geqslant \min S = 1 \implies x \geqslant 1 + \frac{1}{k}$$

Now we proceed with the inductive step. Take $x > G_n$. This implies that

$$k(x-1) > k(G_n - 1) = G_{n-1}$$

Obviously, $k(x-1) \in S$. However, by the induction hypothesis, it follows that $k(x-1) \geqslant G_n$ which rearranges into

$$x \geqslant \frac{1}{k}(G_n + k) = G_{n+1}$$

so the lemma is proved by mathematical induction. □

1 point.

Let $T = \{G_0, G_1, G_2, \ldots\}$. Assume that exists some $a \in S \setminus T$. Then using Lemma 3 we get that

$$a > G_n \text{ and } a \notin T \implies a \geqslant G_{n+1} \text{ and } a \notin T \implies a > G_{n+1}$$

and as $a \neq G_0 = 1 = \min S$, then $a \geqslant \sup T = \frac{k}{k-1}$.

1 point.

However, $a \leqslant \frac{k}{k-1}$ holds as a consequence of Lemma 2, so the only member of $S \setminus T$ is $\frac{k}{k-1}$. Therefore,

$$S \subseteq \left\{\frac{k}{k-1}, G_0, G_1, G_2, \ldots\right\}$$

1 point.

However, if for some $n > 1$, $G_n \in S$, then $G_{n-1} = k(G_n - 1) \in S$, so we have that

$$k(G_n - G_{n-1}) = \frac{1}{k^{n-1}} \in S$$

which is impossible due to $k > 1$, so we in fact have

$$S \subseteq \left\{1, \frac{k+1}{k}, \frac{k}{k-1}\right\}$$

and due to $|S| \geqslant 3$ all three numbers must belong to the set (easy to see that they are distinct). However, then

$$k\left(\frac{k}{k-1} - \frac{k+1}{k}\right) = \frac{1}{k-1} \in \left\{1, \frac{k+1}{k}, \frac{k}{k-1}\right\}$$

which gives $k \in \left\{\frac{1+\sqrt{5}}{2}, 2\right\}$, both of which satisfy the condition by verification.

1 point.

Second Solution. Verification is the same and also worth **1 point**. For a set $A \subseteq \mathbb{R}^+$, we will write $\triangle A = \{a - b \mid a, b \in A, a > b\}$. Suppose $k > 1$ is such that there exists a set S with the required properties.

Lemma 1. If $d \in \triangle S$ is not a maximal element, then $kd \in \triangle S$.

Proof. Let $a, b \in S$ be such that $a - b = d > 0$. Since d is not maximal in $\triangle S$, either a is not maximal in S or b is not minimal in S. If the former is true, then $\exists c \in S$ with $c > a$, hence $k(c-a), k(c-b) \in S$. But then $k(c-b) - k(c-a) = k(a-b) = kd \in \triangle S$, as desired. Otherwise, $\exists c \in S$ with $c < b$, so $k(b-c), k(a-c) \in S$, hence $k(a-c) - k(b-c) = k(a-b) = kd$, so we are done. ☐

2 points.

Lemma 2. $\triangle S$ is a finite geometric progression with common ratio k. In particular, S is finite.

Proof. First note that $\triangle S$ must have a maximal element M. Indeed, otherwise we could take $d \in \triangle S$ and inductively obtain $k^n d \in \triangle S$ for all $n \in \mathbb{N}$, which is absurd since $\triangle S$ is bounded as S is bounded.

1 point.

Now for any $d \in \triangle S$, take the maximal $n \in \mathbb{N}_0$ such that $k^n d \leqslant M$. Then it follows inductively that $k^i d \in \triangle S$ for $0 \leqslant i \leqslant n$. By maximality of n, $k^{n+1} d > M$, so we must have $k^n d = M$ (otherwise we would have $k^{n+1} d \in \triangle S$ by the Lemma 1). It follows that $d = \frac{M}{k^n}$ and also $\frac{M}{k^i} \in \triangle S$ for all $0 \leqslant i < n$. Hence, $\triangle S$ is a (possibly infinite) geometric progression with common ratio $\frac{1}{k}$.

2 points.

Suppose that $\triangle S$ is infinite. Then S contains an infinite geometric progression with ratio $\frac{1}{k}$. Then for any $a, b \in S$ with $a > b$, one can choose c in this progression with $c < b$, so that $a - c > a - b$. This contradicts the fact that $\triangle S$ has a maximal element, so $\triangle S$ must be finite. ☐

1 point.

Now by scaling WLOG assume that $\triangle S = \{1, k, \ldots, k^{m-1}\}$ for some $m \in \mathbb{N}$. Then $\{k, k^2, \ldots, k^m\} \subseteq S$, hence $\triangle\{k, k^2, \ldots, k^m\} \subseteq \triangle S$. But note that $k^{i+1} - k^i < k^{i+2} - k^{i+1}$ for all $1 \leqslant i < m-1$ and $k^m - k^i > k^m - k^{i+1}$ for all $1 \leqslant i < m-1$, so it follows that $|\triangle\{k, k^2, \ldots, k^m\}| \geqslant 2m - 3$. Hence, $2m - 3 \leqslant m$, i.e. $m \leqslant 3$.

1 point.

Now $m \geqslant |S|-1$, so $|S| \leqslant 4$. If $|S|=4$, then $m=3$ and it can easily be checked that S is an arithmetic progression, say with difference $d>0$. But then $\triangle S = \{d, 2d, 3d\}$, which is not a geometric progression. Hence, $|S|=3$.

1 point.

Now we can write $S=\{a,b,c\}$, with $a<b<c$. As $k(b-a), k(c-b) < k(c-a)$ and $k\triangle S \subseteq \{a,b,c\}$, five cases arise:

- If $k(b-a)=a$, $k(c-b)=a$ and $k(c-a)=b$. Then $\frac{k+1}{k}a = b = k(c-a) = k(c-b)+k(b-a) = 2a$, so $k=1$. ✗
- If $k(b-a)=a$, $k(c-b)=a$ and $k(c-a)=c$. Then $\frac{k}{k-1}a = c = k(c-a) = k(c-b)+k(b-a) = 2a$, so $k=2$. ✓
- If $k(b-a)=a$, $k(c-b)=b$ and $k(c-a)=c$. Then $\frac{k+1}{k}a = b = \frac{k}{k+1}c = \frac{k^2}{(k+1)(k-1)}a$, so $k=\frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$. ✓ or ✗
- If $k(b-a)=b$, $k(c-b)=a$ and $k(c-a)=c$. Then $b=\frac{k}{k-1}a=c$, which is impossible. ✗
- If $k(b-a)=b$, $k(c-b)=b$ and $k(c-a)=c$. Then $\frac{k+1}{k}b = c = k(c-a) = k(c-b)+k(b-a) = 2b$, so $k=1$. ✗

1 point.

Third Solution. Verification is the same and also worth **1 point**. We use the same notation as in the **Second Solution.**

Lemma 1. S is finite.

Proof. Let $m=\inf S$, $M=\sup S$ (these exist since S is bounded both below and above as a subset of $\mathbb{R}$). Then note that $\sup \triangle S = M-m$. This holds since for any $a,b\in S$ we have $a-b\leqslant M-m$ and moreover given any $\varepsilon>0$, there exist $a,b\in S$ such that $a>M-\frac{\varepsilon}{2}$, $b<m+\frac{\varepsilon}{2}$, so that $a-b>M-m-\varepsilon$.

1 point.

Since $k\triangle S\subseteq S$, we have $\sup(k\triangle S)\leqslant M$, i.e. $\sup\triangle S\leqslant \frac{M}{k}$, $M-m\leqslant\frac{M}{k}$, $m\geqslant\frac{k-1}{k}M$.

Again since $k\triangle S\subseteq S$, we have $\inf(k\triangle S)\geqslant m$, i.e. $\inf\triangle S\geqslant\frac{m}{k}\geqslant\frac{k-1}{k^2}M$.

1 point.

So if $a_1,a_2,\ldots,a_n$ are some elements of S with $m\leqslant a_1<a_2<\ldots<a_n\leqslant M$, we have $a_{i+1}-a_i\geqslant\frac{k-1}{k^2}M$ for all $1\leqslant i<n$, so we get

$$\frac{M}{k}\geqslant M-m\geqslant a_n-a_1=\sum_{i=1}^{n-1}a_{i+1}-a_i\geqslant(n-1)\cdot\frac{k-1}{k^2}M,$$

hence $n\leqslant\frac{2k-1}{k-1}$. In particular, S is finite. □

1 point.

Lemma 2. $|S|=3$.

Proof. Let $a_1<a_2<\ldots<a_n$ be the elements of S, and assume for the sake of contradiction that $|S|\geqslant 4$.

We know $k(a_n-a_1)>k(a_{n-1}-a_1)>\ldots>k(a_2-a_1)$ are elements of S, and there are at least $n-2$ elements of S greater than $k(a_2-a_1)$. This implies $k(a_2-a_1)\in\{a_1,a_2\}$. Using a similar argument, $k(a_3-a_1)\in\{a_2,a_3\}$, $k(a_3-a_2)\in\{a_1,a_2\}$ and $k(a_4-a_1)\in\{a_3,a_4\}$.

2 points.

If $k(a_2-a_1)=a_2$, then $k(a_3-a_1)=a_3$, so $a_2=a_1\frac{k}{k-1}=a_3$, which is impossible, therefore $k(a_2-a_1)=a_1$ which implies that $a_2=a_1(1+\frac{1}{k})$.

1 point.

If $k(a_3-a_1)=a_2$, then $a_3=a_1+\frac{a_2}{k}=a_1(1+\frac{1}{k}+\frac{1}{k^2})$, so $k(a_3-a_2)=ka_1(1+\frac{1}{k}+\frac{1}{k^2}-1-\frac{1}{k})=\frac{a_1}{k}<a_1$, which is impossible. Therefore, $k(a_3-a_1)=a_3$ which implies that $a_3=a_1\frac{k}{k-1}$.

1 point.

Now, because $k(a_4-a_1)>k(a_3-a_1)=a_3$, we know that $k(a_4-a_1)=a_4$ as there are $n-4$ differences greater than this, but this implies $a_4=a_1\frac{k}{k-1}=a_3$, a contradiction. Therefore, $|S|=3$. □

1 point.

Similar finish as in the **Second Solution** which is also worth **1 point.**

Alternative proof of Lemma 2.

Fact. Let $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$ and $n \geqslant 3$ be a finite set of real numbers such that $|\triangle A| \leqslant |A|$. Then either

- there exist $j \in \{1, \ldots, n-1\}$ and $0 < d \leqslant a_{j+1} - a_j$ such that $a_{i+1} - a_i = d$ for all $1 \leqslant i < n$ with $i \neq j$ or
- $a_2 - a_1 = a_n - a_{n-1}$ and there exists $0 < d < a_2 - a_1$ such that $a_{i+1} - a_i = d$ for all $1 < i < n-1$.

Proof. Take $j \in \{1, \ldots, n-1\}$ that maximizes $a_{j+1} - a_j$. Suppose first that j can be taken so that $1 < j < n-1$. If $a_{t+1} - a_t = a_{j+1} - a_j$ for all $1 \leqslant t < n$, then we are done, so suppose $\exists t \in \{1, \ldots, n-1\}$ such that $a_{t+1} - a_t < a_{j+1} - a_j$.

Now call a sequence of pairs of indices $(l_1, r_1), (l_2, r_2), \ldots, (l_{n-1}, r_{n-1})$ a *path* if $(l_1, r_1) = (j, j+1)$ and $(l_{i+1}, r_{i+1}) \in \{(l_i, r_i + 1), (l_i - 1, r_i)\}$ for all $1 \leqslant i < n-1$. Define the *signature* of a path to be the sequence $(a_{r_i} - a_{l_i})_{1 \leqslant i \leqslant n-1}$.

We claim that any two paths have the same signature. Indeed, note that for any path, $a_{t+1} - a_t, a_{r_1} - a_{l_1}, a_{r_2} - a_{l_2}, \ldots, a_{r_{n-1}} - a_{l_{n-1}}$ is a strictly increasing sequence of n elements of $\triangle A$, so the elements of the signature are fixed since $|\triangle A| \leqslant n$.

Now given any $p < j, q > j$, we can choose two paths (l_i, r_i) and (l'_i, r'_i) such that $(l_{q-p}, r_{q-p}) = (p, q)$ and $(l'_{q-p}, r'_{q-p}) = (p+1, q+1)$. By the previous observation, it follows that $a_q - a_p = a_{q+1} - a_{p+1}$, i.e. $a_{p+1} - a_p = a_{q+1} - a_q$. Since $1 < j < n-1$, it follows that $a_{q+1} - a_q = a_2 - a_1$ for all $q > j$ and also $a_{p+1} - a_p = a_n - a_{n-1}$ for all $p < j$. Since $a_2 - a_1 = a_n - a_{n-1}$, we have $a_{i+1} - a_i = a_2 - a_1$ for all $i \neq j$, as desired.

It remains to deal with the case when $a_{i+1} - a_i < a_{j+1} - a_j$ for $1 < i < n-1$. Note that $|\{a_{i+1} - a_i \mid 1 \leqslant i < n\}| \leqslant 2$ since otherwise we could choose $1 \leqslant s, t < n$ and a path (l_i, r_i) such that $a_{s+1} - a_s, a_{t+1} - a_t, a_{r_1} - a_{l_1}, a_{r_2} - a_{l_2}, \ldots, a_{r_{n-1}} - a_{l_{n-1}}$ is a strictly increasing sequence of $n+1$ elements of $\triangle A$, which is absurd. The claim now follows. □

3 points.

Now we proceed by proving $|S| = 3$. Suppose for the sake of contradiction that $|S| \geqslant 4$. Enumerate S as $x_1 < x_2 < \ldots < x_n$, where $n \geqslant 4$, Since S satisfies the hypothesis of the lemma, we may consider the following cases:

Case 1. (x_i) is an arithmetic sequence
Let d be the difference of (x_i). Then the enumeration of $k\triangle S$ is an arithmetic subsequence of (x_i) of length $n-1$, with difference kd. Since $n \geqslant 4$, it is either $x_1, \ldots, x_{n-1}$ or $x_2, \ldots, x_n$, so it must have difference d, contradiction.

Case 2. $\exists a, b > 0, j \in \{1, \ldots, n-1\}$ such that $a < b, x_{j+1} - x_j = b$ and $x_{i+1} - x_i = a$ for $1 \leqslant i < n, i \neq j$
Then $k\triangle S = \{ka, kb, k(b+a), \ldots, k(b+(n-2)a)\}$, where $ka < kb < k(b+a) < \ldots < k(b+(n-2)a)$. Hence, $x_2 - x_1 = k(b-a)$ and $x_{i+1} - x_i = ka$ for $1 < i < n$. It follows that $j = 1$, $k(b-a) = b$ and $ka = a$, which is absurd since $k > 1$.

Case 3. $\exists a, b > 0$ such that $a < b, x_2 - x_1 = x_n - x_{n-1} = b$ and $x_{i+1} - x_i = a$ for $1 < i < n-1$
Then $k\triangle S = \{ka, kb, k(b+a), \ldots, k(b+(n-3)a), k(2b+(n-3)a)\}$, where $ka < kb < \ldots < k(b+(n-3)a) < k(2b+(n-3)a)$. Hence, $x_n - x_{n-1} = kb$, which is absurd since $k > 1$. □

2 points.

Notes on marking:

- A student cannot be awarded with points from two different solutions.
- In all solutions, if a student states that verification "is trivial" it should be awarded **0 points.** However, it is enough to give examples of sets for two possible values of k and then the student should be awarded **1 point.** This point can be awarded even if student hasn't solved the problem completely.
- In **First Solution**, if a student writes explicitly that $a \leqslant \frac{k}{k-1}$ without showing that $S \subseteq \left\{\frac{k}{k-1}, G_0, G_1, G_2, \ldots\right\}$ it should also be awarded **1 point.**
- In **Second Solution**, if a student states that deduction from $|S| = 3$ to $k = \frac{1+\sqrt{5}}{2}$ or 2 "is trivial" it should be awarded **0 points.**
- In **Third Solution**, if a student states that $|\triangle A| = |A| - 1$ iff A is an aritmetic sequence, it should be awarded **1 point.** However, if a student states just that $|\triangle A| \geqslant |A| - 1$ for all sequences, it should be awarded **0 points.**
- In **Alternative proof of Lemma 2**, if a student states correctly the whole class of sequences satisfying $|\triangle A| = |A|$, it should be awarded **1 point**.
- If student's solution is true with fact that S is finite, it should be awarded at most **7 points.**
- If student proves that $|S| \leqslant c$ for some $c \in \mathbb{N}$ independent of k, it should be awarded **5 points**) (**1 point** for verification is not included and can also be awarded separetly).

Problem 4. Let x, y, m, n be integers greater than 1 such that

$$\underbrace{x^{x^{x^{\cdot^{\cdot^{x}}}}}}_{m \text{ times}} = \underbrace{y^{y^{y^{\cdot^{\cdot^{y}}}}}}_{n \text{ times}}.$$

Does it follow that $m = n$?

Remark: This is a tetration operation, so we can also write ${}^m x = {}^n y$ for the initial condition.

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Solution. Yes, it does. Assume for the sake of contradiction that $x < y$. Then $m > n$. Define function f recursively

$$f(r) = \begin{cases} f(\log_x(r)) + 1 & \text{if } \log_x(r) \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

for example, if $x = 2$, then $f(256) = f\left(2^{2^3}\right) = 2$. Essentially it is the least possible height of an exponent different from x.

Lemma 1. $f(y) \geqslant 1$.

Proof. Let p be a prime number such that $p \mid x$ (it exists as $x > 1$). Then $p \mid y$, so write $x = p^a \cdot x'$ and $y = p^b \cdot y'$, where $p \nmid x', y'$. Let $a' = \frac{a}{(a,b)}$ and $b' = \frac{b}{(a,b)}$. Let $v_p(r)$ denote the largest integer such that $p^{v_p(r)} \mid r$. Then

$$v_p\left({}^m x\right) = v_p\left({}^n y\right) \implies v_p\left((p^a)^{{}^{m-1}x}\right) = v_p\left(\left(p^b\right)^{{}^{n-1}y}\right) \implies a \cdot {}^{m-1}x = b \cdot {}^{n-1}y \implies x^{a \cdot {}^{m-1}x} = x^{b \cdot {}^{n-1}y} \implies$$

$$\implies \left({}^m x\right)^a = x^{b \cdot {}^{n-1}y} \implies \left({}^n y\right)^a = x^{b \cdot {}^{n-1}y} \implies y^{a \cdot {}^{n-1}y} = x^{b \cdot {}^{n-1}y} \implies y^a = x^b \implies y^{a'} = x^{b'}$$

so there exists z such that $x = z^{a'}$ and $y = z^{b'}$.

2 points.

As $1 \leqslant a' < b'$ and $(a', b') = 1$, then

$$a \cdot {}^{m-1}x = b \cdot {}^{n-1}y \implies \frac{b'}{a'} = \frac{b}{a} = \frac{{}^{n-1}y}{{}^{m-1}x} = \frac{\left(z^{b'}\right)^{{}^{n-2}y}}{\left(z^{a'}\right)^{{}^{m-2}x}} = z^{b' \cdot {}^{n-2}y - a' \cdot {}^{m-2}x} \implies a' \mid b' \implies a' = 1 \implies y = x^{b'}$$

so we conclude that $f(y) \geqslant 1$. □

1 point.

Lemma 2. $f\left({}^n y\right) \leqslant 2$.

Proof. We have two cases depending on $f(y)$.

Case 1. $f(y) = 1$

Write $y = x^k$ where $f(k) = 0$. Then

$$f\left({}^n y\right) = f\left(\left(x^k\right)^{{}^{n-1}y}\right) = f\left(x^{k \cdot {}^{n-1}y}\right) = f\left(k \cdot {}^{n-1}y\right) + 1 = f\left(k \cdot x^{k \cdot {}^{n-2}y}\right) + 1 = 1$$

as if $k \cdot x^{k \cdot {}^{n-2}y} = x^l \implies f(k) = 1$ or $k = 1$ which is impossible, so $f\left(k \cdot x^{k \cdot {}^{n-2}y}\right) = 0$.

3 points.

Case 2. $f(y) \geqslant 2$

Write $y = x^{x^k}$. Then

$$f\left({}^n y\right) = f\left(\left(x^{x^k}\right)^{{}^{n-1}y}\right) = f\left(x^{x^k \cdot {}^{n-1}y}\right) = f\left(x^k \cdot {}^{n-1}y\right) + 1 = f\left(x^k \cdot x^{x^k \cdot {}^{n-2}y}\right) + 1 = f\left(k + x^k \cdot {}^{n-2}y\right) + 2 = 2$$

as if $k + x^k \cdot {}^{n-2}y = x^l \implies x^k \mid k$ which is impossible, so $f\left(k + x^k \cdot {}^{n-2}y\right) = 0$. □

3 points.

Using this conlusion we have that

$$2 \leqslant m = f\left({}^m x\right) = f(LHS) = f(RHS) = f\left({}^n y\right) \leqslant 2 \implies m = 2 \implies x^x < y^y = {}^2 y \leqslant {}^n y = x^x$$

which is impossible, so we conclude $m = n$.

1 point.