6^{TH} European Mathematical Cup

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Senior Category

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Problems and Solutions

Problem 1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that the inequality

$$f(x) + yf(f(x)) \leq x(1 + f(y))$$

holds for all positive integers x, y.

(Adrian Beker)

Solution. We claim that $f(x) = x$ is the only function that satisfies the inequality for all positive integers x, y.
We can see that it is indeed the solution because $x + yx = x(1 + y)$.
Setting $x = 1$ and $y = 1$, we obtain:

$$f(1) + f(f(1)) \leq 1 + f(1),$$

which implies $f(f(1)) \leq 1$, so $f(f(1)) = 1$ because it must be a positive integer.

1 point.

Setting $x = 1$ and $y = f(1)$, we obtain:

$$f(1) + f(1) \leq 1 + f(1),$$

which similarly implies $f(1) = 1$.

3 points.

Now, setting $x = 1$ gives:

$$1 + y \leq 1 + f(y),$$

so $f(y) \geq y$ for all positive integers y.
Setting $y = 1$ and using the previous fact, we write:

$$f(x) + f(f(x)) \leq 2x \leq 2f(x) = f(x) + f(x) \leq f(x) + f(f(x)),$$

so equality holds on each step. In particular, $f(x) = x$ for every positive integer x.

6 points.

Notes on marking:

- Checking that $f(x) = x$ satisfies the inequality is worth **0 points.** If a student shows that a solution, if it exists, must be the identity function ("solves the problem"), but fails to show that the identity function is indeed the solution, he or she shall be deducted **1 point.** A sentence saying something along the lines of "it is trivial to show that the identity function satisfies the inequality", due to the sentence being true, shall not be deducted the point.

Problem 2. A friendly football match lasts 90 minutes. In this problem, we consider one of the teams, coached by Sir Alex, which plays with 11 players at all times.

a) Sir Alex wants for each of his players to play the same integer number of minutes, but each player has to play less than 60 minutes in total. What is the minimum number of players required?

b) For the number of players found in a), what is the minimum number of substitutions required, so that each player plays the same number of minutes?

Remark: Substitutions can only take place after a positive integer number of minutes, and players who have come off earlier can return to the game as many times as needed. There is no limit to the number of substitutions allowed.

(Athanasios Kontogeorgis, Demetres Christofides)

Solution. a) Since exactly 11 players play at all times, the total number of minutes played by all of the players combined is $11 \cdot 90 = 990$. Let n be the number of Sir Alex's players that have participated in the match and let k be the number of minutes which each of them spent playing, with $k < 60$ and $k \in \mathbb{Z}$. Now the equality $nk = 990$ holds.

1 point.

From that fact combined with $k < 60$ we get $n \geqslant 17$ and $n | 990$ as well. Finally, it is easy to conclude that the minimal such n is 18.

1 point.

Construction.

1 point.

b) We can formulate the problem by using graphs. Let us construct a graph with 18 vertices that represent the players. Two vertices are connected by an edge if one of the corresponding players substituted the other.

1 point.

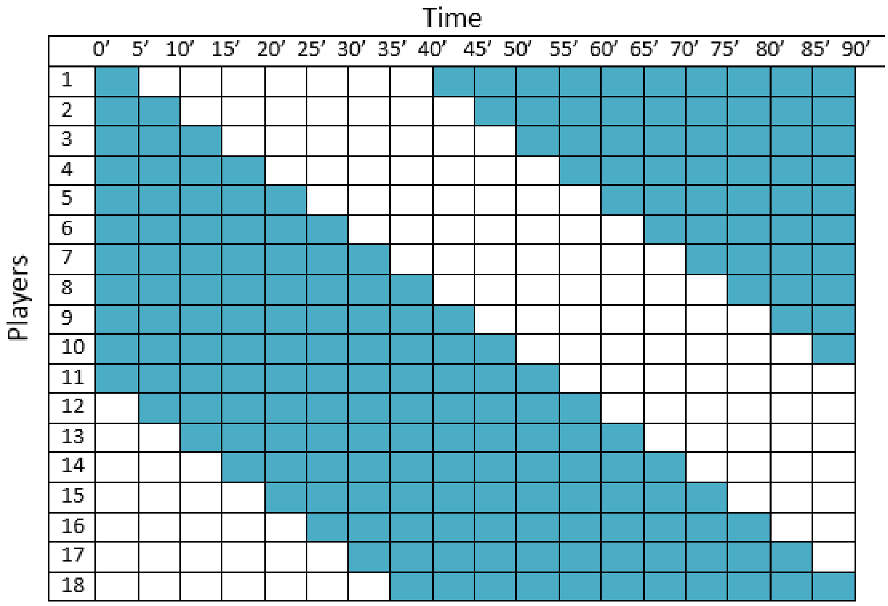
Suppose that less than 17 substitutions were made. Then the graph isn't connected and the smallest connected component consists of $k \leqslant 9$ players among which all of their substitutions were made.

1 point.

Let us suppose that exactly r of them are on the pitch at all times. It is easy to determine that each of the 18 players will play exactly 55 minutes. So the players from the smallest connected component will spend the combined total of $55k$ minutes playing. But, from the same conclusion as earlier, we get the equality $55k = 90r$. It follows that $11 | r$ which implies $r > 9$ and so we reach a contradiction. $\Rightarrow$ The graph is connected and at least 17 substitutions are required.

3 points.

The following table shows a match in which each of the 18 players played 55 minutes and exactly 17 substitutions were made (the shaded regions correspond to the time intervals played by each player).



2 points.

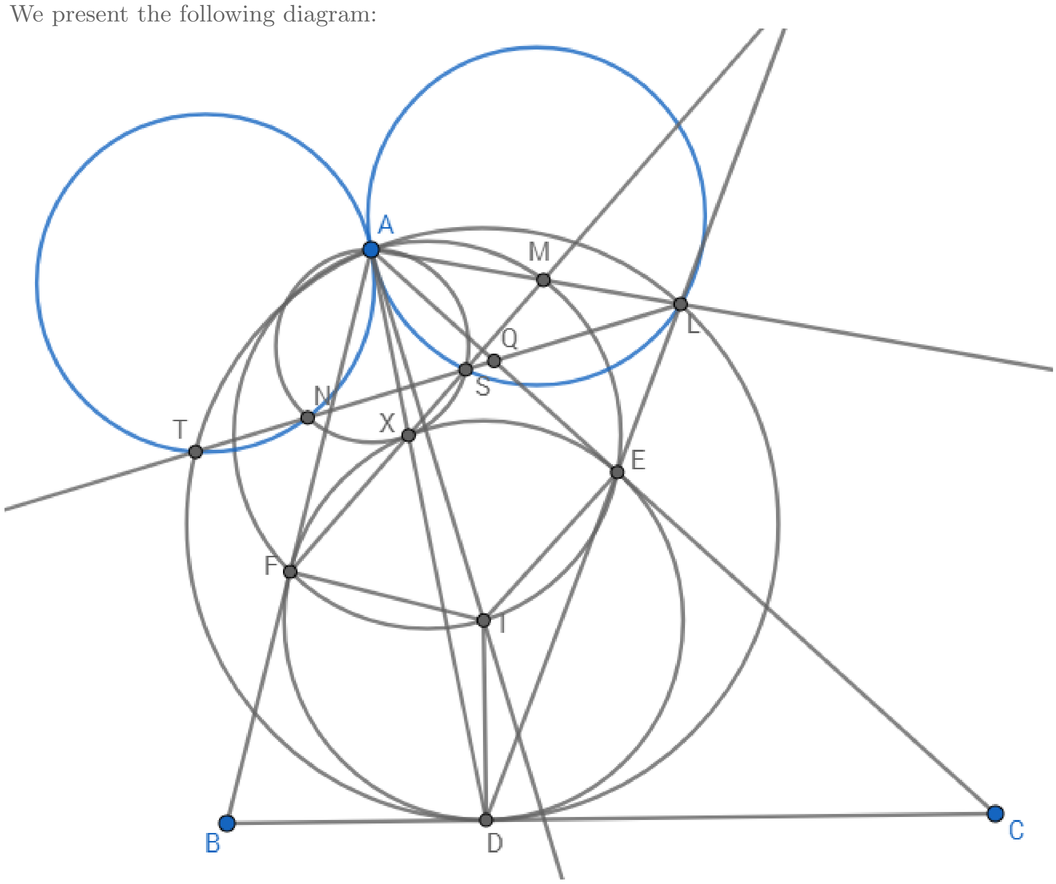
Notes on marking:

- An example for construction in **a)** is the construction from **b)** but there are far easier examples than that and any correct one can bring that 1 point. If a student didn't make the construction in **a)** but finds one in **b)**, he or she shall be awarded both the 1 point from **a)** and 2 points from **b)** for it but if the student doesn't write a construction neither in **a)** nor **b)**, all 3 mentioned points are to be deducted.
- There can be an argument made for **b)** without observing graphs and has to be evaluated accordingly. If a student reaches the conclusion equivalent to the smallest connected component, **2 points** have to be given, one for that conclusion and one that is intended for observing the graph in the official solution.

Problem 3. Let ABC be a scalene triangle and let its incircle touch sides BC, CA and AB at points D, E and F respectively. Let line AD intersect this incircle at point X. Point M is chosen on the line FX so that the quadrilateral $AFEM$ is cyclic. Let lines AM and DE intersect at point L and let Q be the midpoint of segment AE. Point T is given on the line LQ such that the quadrilateral $ALDT$ is cyclic. Let S be a point such that the quadrilateral $TFSA$ is a parallelogram, and let N be the second point of intersection of the circumcircle of triangle ASX and the line TS. Prove that the circumcircles of triangles TAN and LSA are tangent to each other.

(Andrej Ilievski)

Solution. We present the following diagram:



0 points.

Let P be the midpoint of segment AF and let R be the second intersection of EX and the circumcircle of $\triangle AFE$. Let K denote the intersection of lines AR and DF.

By the tangent-chord theorem, we have $\angle EDX = \angle AEX$. Since $DEXF$ is cyclic, we have $\angle EDX = \angle EFX = \angle EFM$. Since $AFEM$ is cyclic, we have $\angle EFM = EAM = \angle EAL$. Hence, $\angle AEX = EAL$, so $AL \parallel EX$, and analogously $AK \parallel FX$. Furthermore, we have $\angle EDX = \angle EAL$, i.e. $\angle EDA = \angle EAL$, so by the converse of the tangent-chord theorem, LA is tangent to the circumcircle of $\triangle AED$.

2 points.

By power of a point, we have $LA^2 = LE \cdot LD$. If we denote the radical axis of the incircle of $\triangle ABC$ and the degenerate circle A by ℓ, this means that L lies on ℓ. Analogously, K lies on ℓ.

1 point.

Now, since $QA^2 = QE^2$, $PA^2 = PF^2$ and QE, PF are tangents to the incircle of $\triangle ABC$, it follows that P, Q both lie on ℓ. Since PQ is the midline of $\triangle AEF$, we have $\ell \parallel EF$.

1 point.

Since $AL \parallel EX$, $AK \parallel FX$ and $XFDE$ is cyclic, it follows that $AKDL$ is also cyclic. Hence, K is the second intersection of LQ and the circumcircle of $\triangle ALD$, so we must have $K \equiv T$, i.e. T, F, D are collinear.

1 point.

Since $TFSA$ is a parallelogram, TS bisects the segment AF, i.e. T, P, S are collinear, which means that S lies on ℓ. Moreover, since $AT \parallel FS$ and $AT \parallel FX$, it follows that F, X, S are collinear.

1 point.

Then since $ANXS$ is cyclic, $\angle NAX = \angle NSX$. Since $NS \parallel FE$, $\angle NSX = \angle EFX$. Since $ALDT$ and $XEDF$ are both cyclic, we have $\angle NTA = \angle LTA = \angle LDA = \angle EDX = \angle EFX$, so $\angle NAX = \angle NTA$. Hence, by the converse of the tangent-chord theorem, AX is tangent to the circumcircle of $\triangle TAN$.

2 points.

Finally, since $AS \parallel TF$, i.e. $AS \parallel FD$, we have $\angle XAS = \angle XDF$. Again, using the cyclicity of $ALDT$ and $XEDF$, we have $\angle ALS = \angle ALT = \angle ADT = \angle XDF$, so $\angle XAS = \angle ALS$. Hence, by the converse of the tangent-chord theorem, AX is tangent to the circumcircle of $\triangle LSA$.

2 points.

Since AX is the common tangent of the circumcircles of triangles TAN and LSA, it follows that they are tangent to each other at A, as desired.

Notes on marking:

- There are many different ways to finish the solution once the collinearities of T, F, D and F, X, S are established. One can, for example, show that E, X, N are also collinear by noting that $\angle XNS = \angle XAS = \angle XDF = \angle XEF$ and using the fact that $SN \parallel FE$, for which a student should be awarded **2 points.** Then one can establish the result by introducing the tangent to the circumcircle of $\triangle TAN$ at A and using the tangent-chord theorem and its converse together with the fact that $\angle NTA + \angle ALS = \angle NAS$ holds. This part is worth **2 points.** However, points from different approaches are not additive, a student should be awarded the maximum of points obtained from one of them.

Problem 4. Find all polynomials P with integer coefficients such that $P(0) \neq 0$ and

$$P^n(m) \cdot P^m(n)$$

is a square of an integer for all nonnegative integers n, m.

Remark: For a nonnegative integer k and an integer n, $P^k(n)$ is defined as follows: $P^k(n) = n$ if $k = 0$ and $P^k(n) = P(P^{k-1}(n))$ if $k > 0$.

(Adrian Beker)

Solution. Let $Q(n, m)$ denote the assertion "$P^n(m) \cdot P^m(n)$ is a square of an integer".
We claim that $P(x) = x + 1$ is the unique polynomial with integer coefficients such that $P(0) \neq 0$ and $Q(n, m)$ is true for all $n, m \in \mathbb{N}_0$.

First we check that this polynomial indeed satisfies the conditions. An easy induction on k shows that $P^k(n) = n + k$ for all $n, k \in \mathbb{N}_0$. Then $P^n(m) \cdot P^m(n) = (m + n)^2$, which is clearly a square of an integer, hence $Q(n, m)$ is true for all $n, m \in \mathbb{N}_0$.

1 point.

Now we show that $P(x) = x + 1$ is the only polynomial satisfying all the conditions.
Consider the sequence $(a_n)_{n \geqslant 0}$ defined by $a_n = P^n(0)$ for all $n \geqslant 0$. Then $Q(n, 0)$ implies that $n \cdot a_n$ is a square of an integer for all $n \in \mathbb{N}_0$.

1 point.

Lemma 1. For all sufficiently large primes p, the sequence (a_n) modulo p is periodic with minimal period of length exactly p. In particular, for all sufficiently large primes p, P is bijective when considered modulo p.

Proof: Fix a prime $p > \max\{|P(0)|, 2\}$. Let t be the smallest positive integer for which there exists a nonnegative integer $s < t$ such that $a_s \equiv a_t \pmod{p}$, such a t exists by the Pigeonhole principle. Then the sequence (a_n) modulo p is eventually periodic with minimal period $a_s, \ldots, a_{t-1}$.

1 point.

Suppose that $t - s < p$ holds, i.e. the length of the period is less than p. Note that there exists $r \in \{s, \ldots, t-1\}$ such that $a_r \not\equiv 0 \pmod{p}$ since otherwise we would have $P(0) \equiv 0 \pmod{p}$. Now let n be an arbitrary nonnegative integer. Then take a positive integer k such that $n + kp \geqslant s$ and $n + kp \equiv r \pmod{t-s}$, such a k exists since p and $t - s$ are relatively prime.

We know that $(n + kp) \cdot a_{n+kp}$ is a quadratic residue modulo p, i.e. $n \cdot a_r$ is a quadratic residue modulo p since $a_{n+kp} \equiv a_r \pmod{p}$ and $n + kp \equiv n \pmod{p}$. But this is impossible since $n \cdot a_r$ attains all residues modulo p (recall that $a_r \not\equiv 0 \pmod{p}$), and we know there exists a quadratic nonresidue modulo p since $p > 2$.

Finally, we conclude that $t - s = p$ must hold, i.e. the length of the minimal period is p. In particular, P is surjective and hence bijective when considered modulo p.

2 points.

Alternative proof: Again, fix a prime $p > |P(0)|$. Since $p \cdot a_p$ is a perfect square, a_p must be divisible by p. It follows that for all $n \geqslant 0$, $a_{n+p} = P^n(a_p) \equiv P^n(0) \equiv a_n \pmod{p}$, hence (a_n) modulo p is periodic with period of length p.

1 point.

Now suppose there exist $i, j \in \{0, 1, \ldots, p-1\}$ with $i < j$ and $a_i \equiv a_j \pmod{p}$. If we let $l = j - i$, then for each $n \geqslant i$ we have $a_{n+l} = P^{n-j+l}(a_j) \equiv P^{n-i}(a_i) \equiv a_n \pmod{p}$. Then it immediately follows inductively that $a_n \equiv a_{n+kl} \pmod{p}$ for all $k \in \mathbb{N}_0$ and similarly $a_n \equiv a_{n+mp} \pmod{p}$ for all $m \in \mathbb{N}_0$. Since p and l are relatively prime, there exist $k, m \in \mathbb{N}_0$ such that $kl - mp = 1$, so we have $a_n \equiv a_{n+1} \pmod{p}$. It follows that the sequence is eventually constant and thus equal to 0 modulo p, which is a contradiction. Hence, the length of the minimal period is indeed p and we conclude similarly as in the first proof.

2 points.

Lemma 2. The degree of P is at most 1.

Proof: Assume the contrary and consider the polynomial $Q(x) = P(x + 1) - P(x)$. Then Q is a polynomial with integer coefficients and $\deg Q = \deg P - 1 \geqslant 1$, so Q is nonconstant. A well-known fact due to Schur implies that there are infinitely many primes that divide $Q(n)$ for some integer n. So there are infinitely many primes p such that P is not bijective modulo p, contradicting the result of Lemma 1. Hence, the lemma is proved.

3 points.

By Lemma 2, we can write $P(x) = ax + b$ for some $a, b \in \mathbb{Z}$. $Q(1, 0)$ implies that b is a perfect square, so b is a positive integer since $P(0) \neq 0$.

An easy induction shows that $P^k(0) = b(1 + a + \ldots + a^{k-1})$ for all $k \in \mathbb{N}$. $Q(p, 0)$ implies that $pb(1 + a + \ldots + a^{p-1})$ is a perfect square, i.e. $p(1 + a + \ldots + a^{p-1})$ is a perfect square for all primes p. So $1 + a + \ldots + a^{p-1}$ must be divisible by p, but then $(1 + a + \ldots + a^{p-1})(a - 1) = a^p - 1$ is also divisible by p. By Fermat's little theorem, we know that $a^p - 1 \equiv a - 1 \pmod{p}$, hence p divides $a - 1$ for all primes p, so we must have $a = 1$, i.e. $P(x) = x + b$.

1 point.

Finally, $Q(1, 4)$ implies that $4b^2 + 17b + 4$ is a perfect square, but since $(2b+2)^2 < 4b^2 + 17b + 4 < (2b+5)^2$, $4b^2 + 17b + 4$ must be of the form $(2b+k)^2$ for some $k \in \{3, 4\}$. It is easily checked that $b = 1$ is the only possibility, leaving $P(x) = x+1$ as the only solution.

1 point.

Notes on marking:

- The points from different proofs of Lemma 1 are not additive, a student should be awarded the maximum of points obtained from one of them.