6$^{\text{TH}}$ European Mathematical Cup

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Senior Category

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Problem 1. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that the inequality

$$f(x) + yf(f(x)) \leq x(1 + f(y))$$

holds for all positive integers x, y.

(Adrian Beker)

Problem 2. A friendly football match lasts 90 minutes. In this problem, we consider one of the teams, coached by Sir Alex, which plays with 11 players at all times.

a) Sir Alex wants for each of his players to play the same integer number of minutes, but each player has to play less than 60 minutes in total. What is the minimum number of players required?

b) For the number of players found in a), what is the minimum number of substitutions required, so that each player plays the same number of minutes?

Remark: Substitutions can only take place after a positive integer number of minutes, and players who have come off earlier can return to the game as many times as needed. There is no limit to the number of substitutions allowed.

(Athanasios Kontogeorgis, Demetres Christofides)

Problem 3. Let ABC be a scalene triangle and let its incircle touch sides BC, CA and AB at points D, E and F respectively. Let line AD intersect this incircle at point X. Point M is chosen on the line FX so that the quadrilateral $AFEM$ is cyclic. Let lines AM and DE intersect at point L and let Q be the midpoint of segment AE. Point T is given on the line LQ such that the quadrilateral $ALDT$ is cyclic. Let S be a point such that the quadrilateral $TFSA$ is a parallelogram, and let N be the second point of intersection of the circumcircle of triangle ASX and the line TS. Prove that the circumcircles of triangles TAN and LSA are tangent to each other.

(Andrej Ilievski)

Problem 4. Find all polynomials P with integer coefficients such that $P(0) \neq 0$ and

$$P^n(m) \cdot P^m(n)$$

is a square of an integer for all nonnegative integers n, m.

Remark: For a nonnegative integer k and an integer n, $P^k(n)$ is defined as follows: $P^k(n) = n$ if $k = 0$ and $P^k(n) = P(P^{k-1}(n))$ if $k > 0$.

(Adrian Beker)