5TH European Mathematical Cup

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Junior Category

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Problem 1. A grasshopper is jumping along the number line. Initially it is situated at zero. In k-th step, the length of his jump is k.

- **a)** If the jump length is even, then it jumps to the left, otherwise it jumps to the right (for example, firstly it jumps one step to the right, then two steps to the left, then three steps to the right, then four steps to the left...). Will it visit on every integer at least once?
- **b)** If the jump length is divisible by three, then it jumps to the left, otherwise it jumps to the right (for example, firstly it jumps one step to the right, then two steps to the right, then three steps to the left, then four steps to the right...). Will it visit every integer at least once?

(Matko Ljulj)

Problem 2. Two circles C_1 and C_2 intersect at points A and B. Let P, Q be points on circles C_1, C_2 respectively, such that $|AP| = |AQ|$. The segment $\overline{PQ}$ intersects circles C_1 and C_2 in points M, N respectively. Let C be the center of the arc BP of C_1 which does not contain point A and let D be the center of arc BQ of C_2 which does not contain point A. Let E be the intersection of CM and DN. Prove that AE is perpendicular to CD.

(Steve Dinh)

Problem 3. Prove that for all positive integers n there exist n distinct, positive rational numbers with sum of their squares equal to n.

(Daniyar Aubekerov)

Problem 4. We will call a pair of positive integers (n, k) with $k > 1$ *a lovely couple* if there exists a table $n \times n$ consisting of ones and zeros with following properties:

- In every row there are exactly k ones.
- For each two rows there is exactly one column such that on both intersections of that column with the mentioned rows, number one is written.

Solve the following subproblems:

a) Let $d \neq 1$ be a divisor of n. Determine all remainders that d can give when divided by 6.

b) Prove that there exist infinitely many lovely couples.

(Miroslav Marinov, Daniel Atanasov)