4TH European Mathematical Cup
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Senior Category
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Problems and Solutions

Problem 1. $A = \{a, b, c\}$ is a set containing three positive integers. Prove that we can find a set $B \subset A$, $B = \{x, y\}$ such that for all odd positive integers m, n we have

$$10|x^m y^n - x^n y^m.$$

(Tomi Dimovski)

Solution. Let $f(x, y) = x^m y^n - x^n y^m$. If $n = m$, the problem statement will be fulfilled no matter how we choose B so from now on, without loss of generality, we consider $n > m$. Since m and n are both odd, we have that $n - m$ is even and we get

$$f(x, y) = x^m y^m (y^{n-m} - x^{n-m})$$

$$\implies f(x, y) = x^m y^m (y^2 - x^2) Q(x, y)$$

$$\implies f(x, y) = x^m y^m (y - x)(y + x) Q(x, y),$$

where $Q(x, y) = y^{n-m-2} + y^{n-m-4}x^2 + \cdots + x^{n-m-2}$.

Now if one of x, y is even, $f(x, y)$ is even. If both are odd, then $f(x, y)$ is again even since $x + y$ and $x - y$ are even in that case. This shows that we only need to consider divisibility by 5. If A contains at least one element divisible by 5, we can put it in B and that will give us the solution easily. Now we consider the case when none of the elements in A is divisible by 5. If some two numbers in A give the same remainder modulo 5, we can choose them and then $x - y$ will be divisible by 5 which solves the problem. Now we consider the case when all remainders modulo 5 in A are different. Take a look at the pairs $(1, 4)$ and $(2, 3)$. Since we have three different remainders modulo 5 in A, by pigeonhole principle one of these pairs has to be completely in A (when elements are considered modulo 5). Then if we pick the numbers from A that correspond to those two remainders we get that $x + y$ is divisible by 5 so the problem statement is fulfilled again. This completes the proof.

Problem 2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+b+c+3}{4} \geqslant \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}.$$

(Dimitar Trenevski)

First Solution. Rewrite the left hand side of inequality in following way:

$$\frac{a+b+c+3}{4} = \frac{a+b+c+3}{4\sqrt{abc}} = \frac{a+1}{4\sqrt{abc}} + \frac{b+1}{4\sqrt{abc}} + \frac{c+1}{4\sqrt{abc}}.$$

Rewrite denominators:

$$\frac{a+1}{4\sqrt{abc}} + \frac{b+1}{4\sqrt{abc}} + \frac{c+1}{4\sqrt{abc}} = \frac{a+1}{2\sqrt{ab \cdot c} + 2\sqrt{ac \cdot b}} + \frac{b+1}{2\sqrt{bc \cdot a} + 2\sqrt{ab \cdot c}} + \frac{c+1}{2\sqrt{ac \cdot b} + 2\sqrt{bc \cdot a}},$$

and then by artithmetic mean – geometric mean inequality, we have

$$= \frac{a+1}{2\sqrt{ab \cdot c} + 2\sqrt{ac \cdot b}} + \frac{b+1}{2\sqrt{bc \cdot a} + 2\sqrt{ab \cdot c}} + \frac{c+1}{2\sqrt{ac \cdot b} + 2\sqrt{bc \cdot a}} \geqslant \frac{a+1}{ab+c+ac+b} + \frac{b+1}{bc+a+ab+c} + \frac{c+1}{ac+b+bc+a}.$$

This problem is now solved, because

$$\frac{a+1}{ab+c+ac+b} + \frac{b+1}{bc+a+ab+c} + \frac{c+1}{ac+b+bc+a} = \frac{a+1}{(a+1)(b+c)} + \frac{b+1}{(b+1)(a+c)} + \frac{c+1}{(c+1)(a+b)} =$$

$$= \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$$

Second Solution. We introduce change of variables: $a = x^3$, $b = y^3$, $c = z^3$. We now have the condition $xyz = 1$. We apply Schur inequality (with exponent $r = 1$) to the numerator of the left hand side:

$$x^3 + y^3 + z^3 + 3xyz \geqslant x^2y + x^2z + y^2x + y^2z + z^2x + z^2y,$$

to obtain inequality

$$\frac{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y}{4} \geqslant \frac{1}{x^3+y^3} + \frac{1}{y^3+z^3} + \frac{1}{z^3+x^3}.$$

We apply arithmetic mean – geometric mean inequality for the denominators of the right hand side:

$$x^3 + y^3 \geqslant 2x^{3/2}y^{3/2} \implies \frac{1}{x^3+y^3} \leqslant \frac{1}{2x^{3/2}y^{3/2}} = \frac{1}{2}z^2\sqrt{yz},$$

and similarly to the other terms. We now have to prove

$$\frac{x^2y + x^2z + y^2x + y^2z + z^2x + z^2y}{2} \geqslant x^2\sqrt{yz} + y^2\sqrt{xz} + z^2\sqrt{xy}.$$

We apply arithmetic mean – geometric mean inequality in pairs on the left hand side:

$$\frac{x^2y + x^2z}{2} \geqslant x^2\sqrt{yz},$$

$$\frac{y^2x + y^2z}{2} \geqslant y^2\sqrt{xz},$$

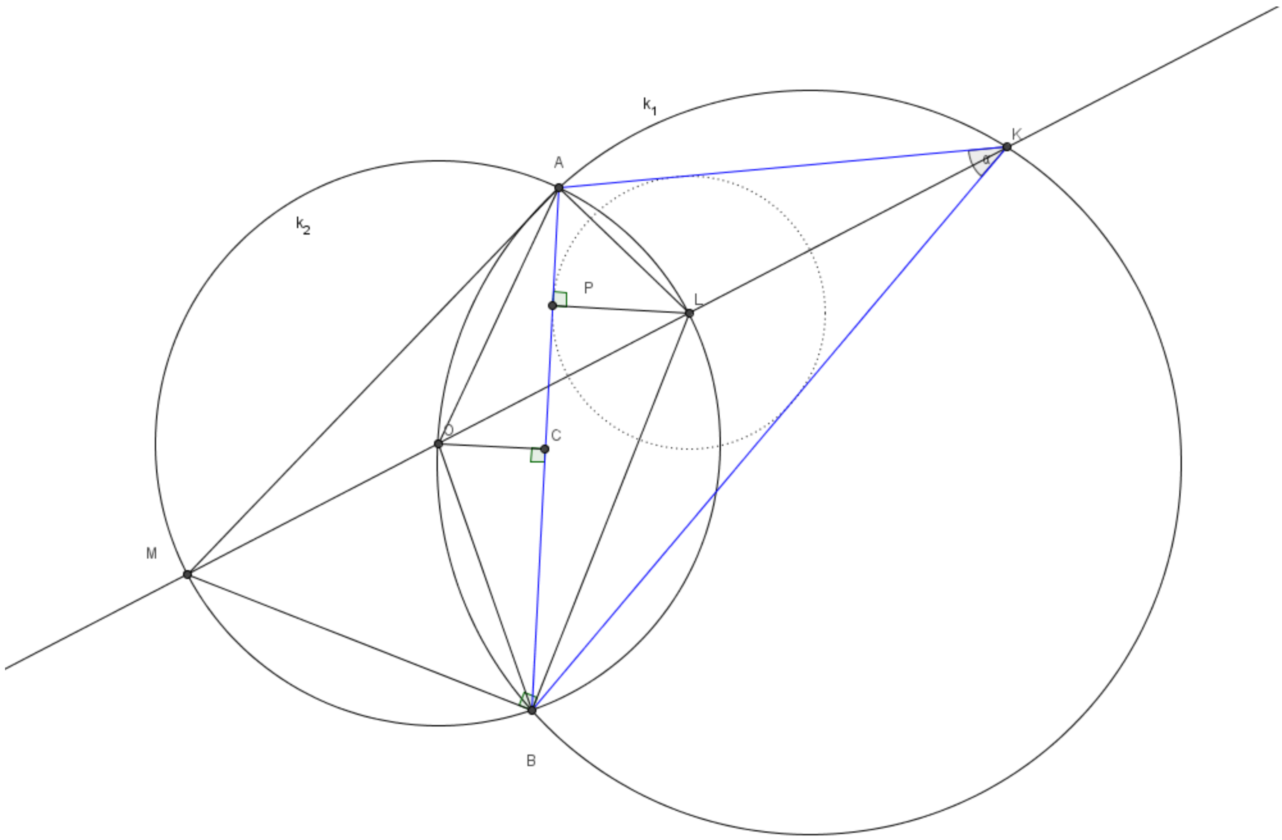
$$\frac{z^2x + z^2y}{2} \geqslant z^2\sqrt{xy}.$$

Summing up inequalities from above finishes the proof.

Problem 3. Circles k_1 and k_2 intersect in points A and B, such that k_1 passes through the center O of the circle k_2. The line p intersects k_1 in points K and O and k_2 in points L and M, such that the point L is between K and O. The point P is orthogonal projection of the point L to the line AB. Prove that the line KP is parallel to the M-median of the triangle ABM.

(Matko Ljulj)

Solution. Let the point C be the midpoint of the line segment $\overline{AB}$. We have to prove $MC \parallel KP$.



Let us introduce angle $\alpha := \angle BKA$. Notice that

$$\angle BLA = 180 - \angle BMA = 180 - \frac{1}{2}\angle BOA = 180 - \frac{1}{2}(180 - \angle BKA) = 90 + \frac{1}{2}\alpha.$$

Also, notice that the point O is midpoint of the arc $\overset{\frown}{AB}$. Thus the line KO is bisector of the angle $\angle BKA$. From the two claims above, we deduce that L is incenter of the triangle ABK. Moreover, notice that $\overline{ML}$ is diameter of the circle k_2, thus $\angle ABM = 90$. Since BL is angle bisector of the angle $\angle ABK$, we deduce that BM is exterior angle bisector of the same angle. Thus, since M lies on angle bi sector KM and exterior angle bisector BM, M is the center of the excircle for the triangle ABK. Thus, we have to prove that the line passing through the incenter L of the triangle ABK and point of the tangency of incircle of the same triangle is parallel to the line passing through the center of the excircle M and the midpoint C of the line segment $\overline{AB}$. This is a well known lemma, which completes the proof.

Problem 4. A group of mathematicians is attending a conference. We say that a mathematician is k-content if he is in the room with at least k people he admires or if he is admired by at least k other people in the room. It is known that when all participants are in the same room then they are all at least $3k+1$-content. Prove that you can assign everyone into one of the 2 rooms in a way that everyone is at least k-content in his room and neither room is empty. *Admiration is not necessarily mutual and no one admires himself.*

(Matija Bucić)

Solution. We will for simplicity and clarity of presentation use some basic graph theoretic terms, this is in no way essential.

We represent the situation by a directed graph (abbr. digraph) $G(V,E)$ where each vertex $v \in V(G)$ represents a mathematician and each edge $e \in E(G)$ represents an admiration relation. Given $v \in V(G)$ we define out-degree of v denoted $o(v)$ as the number of edges starting in v (so the number of mathematicians v admires) and in-degree $i(v)$ as the number of edges ending in v (so the number of mathematicians who admire v). Given $X \subseteq V$ by $G(X)$ we denote the induced subgraph (a graph with vertex set X and edges inherited from G). We say that a digraph is a k-digraph if for every $v \in V(G)$ we have $i(v) \geq k$ or $o(v) \geq k$.

So the question can be reformulated as: Given G is a $3k+1$-digraph we can split its vertices into 2 vertex disjoint classes such that each induced subgraph on class is a k digraph.

We call a subset X of vertices of G *k-tight* if for any $Y \subseteq X$ we have a vertex $v \in Y$ such that $i_{G(Y)}(v) \leq k$ and $o_{G(Y)}(v) \leq k$. A partition of V, (A_1, A_2) is feasible if A_1 is k-tight and A_2 is k-tight.

We first assume there are no feasible partitions.

In this case consider a minimal size subset $A_1 \subseteq V(G)$ subject to $G(A_1)$ being a k-digraph, we define $A_2 \equiv V(G) - A_1$. Given a subset $X \subset A_1$, $G(X)$ is not a k-digraph so there is a vertex $v \in X$ such that $o_{G(X)}(v) < k$ and $i_{G(X)}(v) < k$ which shows that any proper subset of A_1 satisfies the condition of k-tightness. For the case of $X \equiv A_1$ by removing any vertex $v \in A_1$ the graph $G' \equiv G(A_1 - \{v\})$, by minimality assumption on A_1, must contain a vertex w such that $o_{G'}(w) < k$ and $i_{G'}(w) < k$ so as there is only one extra vertex in $G(A_1)$, namely v $o_{G(A_1)}(w) \leq k$, $i_{G(A_1)}(w) \leq k$. In particular this shows A_1 is k-tight.

This implies A_2 is not k-tight by our assumption so there exists an $A_2' \subseteq A_2$ such that A_2' is a $k+1$ digraph. Now applying the following proposition to extend the pair (A_1, A_2') to a full partition which satisfies the conditions of the problem.

Given disjoint subsets $A, B \subseteq V(G)$ we say (A,B) is a solution pair if both $G(A)$ and $G(B)$ are k-digraphs.

Proposition: If a $2k+1$ digraph G admits a solution pair it admits a partition with both induced graphs of both classes being k-digraphs.

Proof. Take a maximal solution pair (A,B), the condition in the lemma guaranteeing it exists. Let $C = V(G) - (A \cup B)$, if C is empty we are done so assume $|C| > 0$. By our assumption $(A, B \cup C)$ is not a solution pair so there is some $x \in C$ such that $o_{G(B\cup C)}(x), i_{G(B \cup C)}(x) < k$ so as G is $2k+1$ digraph $i_G(x) \geq 2k+1$ or $o_G(x) \geq 2k+1$ so either $o_{G(A\cup\{x\})}(x) > k+1$ or $i_{G(A \cup \{x\})}(x) > k+1$ so in particular $(A \cup \{x\}, B)$ is a solution pair contradicting maximality and completing our argument. ■

Hence we are left with the case in which we have at least one feasible partition. We pick the feasible partition (A,B) maximizing $w(A,B) = |E(G(A))| + |E(G(B))|$. The fact that A is k-tight implies there is an x with $o_{G(A)}(x) \leq k$, $i_{G(A)}(x) \leq k$ so x needs to have at least $k+1$ edges in or out of B so $|B| \geq k+1$ and by symmetry $|A| \geq k+1$.

We now prove that there exist an $X \subseteq A$ such that $G(X)$ is a k-digraph, by contradiction. Assuming the opposite we notice that for any $x \in B$, $B - \{x\}$ is still k-tight while B being k-tight implies there is an $x \in B$ such that $o_{G(B)}(x) \leq k$, $i_{G(B)}(x) \leq k$ so for this x we have $A \cup \{x\}$ is also k-tight. Hence, for $A' = A \cup \{x\}$ and $B' = B - \{x\}$, (A', B') is a feasible partition. We considering the change in edges which moving x causes we have $w(A',B') - w(A,B) \geq 3k+1-k-k-k = 1$ as we know $i_G(x) \geq 3k+1$ or $o_G(x) \geq 3k+1$ so moving x from B to A increases number of edges in A by at least $3k+1-k$ while the choice of x in B means we lose at most $k+k$ edges in B. This is a contradiction to maximality of (A,B).

Analogously we can find $Y \subseteq B$ with $G(Y)$ a k-digraph. Now applying the above proposition yet again we are done. ■

Remark: The same argument with slightly modified weight function can be used to show the result for non symmetric rooms, in particular if the graph is a $k + l + \max(k,l) + 1$ digraph it can be partitioned into k- digraph and l digraph parts.