4^{TH} European Mathematical Cup

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Junior Category

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Problem 1. We are given an $n \times n$ board. Rows are labeled with numbers 1 to n downwards and columns are labeled with numbers 1 to n from left to right. On each field we write the number $x^2 + y^2$ where (x, y) are its coordinates. We are given a figure and can initially place it on any field. In every step we can move the figure from one field to another if the other field has not already been visited and if at least one of the following conditions is satisfied:

- the numbers in those 2 fields give the same remainders when divided by n,
- those fields are point reflected with respect to the center of the board.

Can all the fields be visited in case:

a) $n = 4$,
b) $n = 5$?

(Josip Pupić)

Problem 2. Let m, n, p be fixed positive real numbers which satisfy $mnp = 8$. Depending on these constants, find the minimum of

$$x^2 + y^2 + z^2 + mxy + nxz + pyz,$$

where x, y, z are arbitrary positive real numbers satisfying $xyz = 8$. When is the equality attained?
Solve the problem for:

a) $m = n = p = 2$,
b) arbitrary (but fixed) positive real numbers m, n, p.

(Stijn Cambie)

Problem 3. Let $d(n)$ denote the number of positive divisors of n. For positive integer n we define $f(n)$ as

$$f(n) = d(k_1) + d(k_2) + \ldots + d(k_m),$$

where $1 = k_1 < k_2 < \cdots < k_m = n$ are all divisors of the number n. We call an integer $n > 1$ *almost perfect* if $f(n) = n$. Find all almost perfect numbers.

(Paulius Ašvydis)

Problem 4. Let ABC be an acute angled triangle. Let B', A' be points on the perpendicular bisectors of AC, BC respectively such that $B'A \perp AB$ and $A'B \perp AB$. Let P be a point on the segment AB and O the circumcenter of the triangle ABC. Let D, E be points on BC, AC respectively such that $DP \perp BO$ and $EP \perp AO$. Let O' be the circumcenter of the triangle CDE. Prove that B', A' and O' are collinear.

(Steve Dinh)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.