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Senior Category

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Problems and Solutions

Problem 1. In each field of a table there is a real number. We call such $n \times n$ table *silly* if each entry equals the product of all the number in the neighbouring fields.

a) Find all 2×2 silly tables.
b) Find all 3×3 silly tables.

(Two fields of a table are neighbouring if they share a common side.) *(Borna Vukorepa)*

Solution. We solve the subproblems separately.

a) Denote the numbers in the table as on the picture:

a	b
c	d

By the problem condition we have the following:

$$a = bc$$
$$b = ad$$
$$c = ad$$
$$d = bc.$$

From here we can see $a = bc = d$ and $b = ad = c$. When we apply this to the upper relations we get $a = b^2$ and $b = a^2$ and so $a = b^2 = a^4 \iff a(a-1)(a^2+a+1) = 0$. The real solutions to this problem are $a = 0$ and $a = 1$. Now we can see that all 2×2 silly tables are those with all element equal and furthermore equal to zero or one.

b) Denote by a, b, c, d the elements in the table which have exactly three neighbours. We denote the remaining elements in terms of these and get the following table:

ab	a	ad
b	$abcd$	d
bc	c	cd

Let's assume that $abcd = 0$. This implies that the middle element is zero which further implies all its neighbours are zero and consequently every element in the table is zero. And thus only silly table under in this case is all zeros table.
Now assume that $abcd \neq 0$, i.e. none of the table elements is equal to zero. Using the remaining conditions we get:

$$a = (ab)(abcd)(ad) = a^3b^2d^2c \iff a^2b^2d^2c = 1,$$

Analogously we get $a^2b^2c^2d = 1$, $a^2c^2d^2b = 1$ i $b^2c^2d^2a = 1$ (we are allowed to divide by a, b, c, d as they are all non-zero). Equating the LHSs of these equations we get $a = b = c = d$. Inserting this in any of these equations we get $a^7 = 1 \implies a = 1$.
Thus all 3×3 silly tables are all ones and all zeros tables.

Problem 2. *Palindrome* is a sequence of digits which doesn't change if we reverse the order of its digits. Prove that a sequence $(x_n)_{n=0}^{\infty}$ defined as

$$x_n = 2013 + 317n$$

contains infinitely many numbers with their decimal expansions being palindromes.

(Stijn Cambie)

First solution. We will prove the following lemma providing two proofs:

Lema 1. *There is infinitely many numbers divisible with* 317 *with their decimal expansions consisting only of ones.*

Proof. Considering the sequence 1, 11, 111, ... consisting of infinitely many numbers. This numbers have some residues modulo 317. By The Pigeonhole Principle there are at least two numbers in this sequence with the same residue modulo 317. Let the smaller of these two have l digits and larger k. Their difference is

$$\underbrace{111\ldots1}_{k \text{ times}} - \underbrace{111\ldots1}_{l \text{ times}} = \underbrace{111\ldots1}_{(k-l) \text{ times}} \underbrace{000\ldots0}_{l \text{ times}}$$

divisible by 317. It will also remain divisible by 317 if we divide it by 10^l (as 10 and 317 are coprime). This way we get a number consisting only of ones divisible by 317. Let's denote the number of its digits by k. We get infinitely many such numbers by considering numbers consisting of k, $2k$, $3k$, ... ones. □

Proof. As 317 is prime, and as it is coprime with 10 by Fermat's Little Theorem

$$10^{316} \equiv 1 \ (mod\ 317) \implies 317 \mid 10^{316m} - 1, \forall m \in \mathbb{Z}, m \geqslant 1.$$

As 9 is coprime with 317 as well, numbers of the form $\frac{1}{9}(10^{316m} - 1)$, $m \in \mathbb{Z}, m \geqslant 1$ have the property we desire. □

Continuing with the solution we can note that some integer m is in the sequence $(x_n)_{n=0}^{\infty}$ if and only if $m \geqslant 2013$ and $m \equiv 2013 \equiv 111\ (mod\ 317)$. Let $(y_n)_{n=0}^{\infty}$ be a sequence of infinitely many positive integers with their decimal expansions consisting only of ones and each being divisible by 317 (we are using our lemma here). Now numbers

$$1000y_n + 111$$

are in the sequence (as they have the remainder 111 modulo 317) and their decimal expansions are palindromes. Thus there is infinitely many members of the sequence $(x_n)_{n=0}^{\infty}$ whose decimal expansions are palindromes.

Second solution. We will prove the generalised version of the problem for the sequence $(x_n)_{n=0}^{\infty}$ defined as $x_n = a + nb$, where a, b are arbitrary positive integers with the property that b is coprime with 10. The problem is a special case of this for $a = 2013$ i $b = 317$.

We define the sequence $(y_n)_{n=0}^{\infty}$ in the following way: $y_n = 10^{n\varphi(b)}$. Using The Euler's Theorem, $y_n \equiv 1\ (mod\ b)$. Considering the number $1 + y_n + y_n^2 + \ldots y_n^{a-1}$, its decimal expansion is:

$$1 \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}} 1 \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}} \ldots 1 \underbrace{000\ldots0}_{n\varphi(b)-1 \text{ times}} 1$$

where the digit one is repeated a times. It is clear now that the decimal expansion of this number is a palindrome. On the other hand $1 + y_n + y_n^2 + \ldots y_n^{a-1} \equiv 1 + 1 + \ldots 1 = a\ (mod\ b)$, so this number is in the sequence $(x_n)_{n=0}^{\infty}$, for each number n. Thus we have found infinitely many members of the sequence $(x_n)_{n=0}^{\infty}$ with their decimal expansions being palindromes as we wanted.

Problem 3. We call a sequence of n digits one or zero a *code*. Subsequence of a code is a *palindrome* if it is the same after we reverse the order of its digits. A palindrome is called *nice* if its digits occur consecutively in the code. *(Code* (1101) *contains 10 palindromes, of which 6 are nice.)*

a) What is the least number of palindromes in a code?
b) What is the least number of nice palindromes in a code?

(Ognjen Stipetić)

Solution. We will consider the two subproblems separately:

a) Consider any code. Assume there is k digits one and $n - k$ digits zero. We now transform this code into

$$\underbrace{111\ldots1}_{k \text{ puta}}\underbrace{000\ldots0}_{n-k \text{ puta}}$$

by preserving the order among same digits. Lets note that each palindrome consisting of same digits is in the initial code if and only if it is in the transformed code. The transformed code doesn't have a palindrome not consisting of same digits and thus the transformed code has less or equal palindromes than the initial one.

Thus we conclude that it is enough to consider only the codes starting with k digits one and ending in $n - k$ zeros, for some $k \in \{0, 1, \ldots n\}$.

Let us fix a $k \in \{0, 1, \dots n\}$. The code consisting of k ones and $n-k$ zeros has $2^k - 1 + 2^{n-k} - 1 = 2^k + 2^{n-k} - 2$ palindromes. We now seek k which minimizes this expression.

If n is even ($n = 2m$), by the AM-GM inequality $2^k + 2^{n-k} \geqslant 2 \cdot \sqrt{2^{k+n-k}} = 2^m + 2^m \implies$ the least possible number of palindromes in the code with $2m$ digits is $2^m + 2^m - 2 = 2^{m+1} - 2$, and this number is clearly attained for the code with m digits one and ending in m digits zero.

If n is odd ($n = 2m+1$) we have the following inequality for each $k \in \{0, 1, \dots m-1\}$:

$$2^k + 2^{n-k} > 2^{k+1} + 2^{n-k-1} \ (\iff 2^{n-k-1} > 2^k)$$

From this we also get $2^k + 2^{n-k-1} < 2^{k-1} + 2^{n-k+1}$ for all $k \in \{m+1, m+2, \dots 2m+1\}$. So:

$$2^0 + 2^n > 2^1 + 2^{n-1} > \dots > 2^m + 2^{m+1} = 2^{m+1} + 2^m < 2^{m+2} + 2^{m-1} < \dots < 2^n + 2^0$$

Now it is clear that the least number of palindromes in the code with $2m+1$ digits is $2^m + 2^{m+1} - 2$ and this number is attained by the code of m digits one and $m+1$ digits zero.

b) For $n = 1$ we clearly see that the answer is 1. From now on we assume $n \geqslant 2$.

As well for simplicity of the write-up we will not consider the one-digit palindromes as nice as we know that each code of n digits consists of n one-digit palindromes, each of which is nice. So we will find the smallest possible number of multi-digit nice palindromes and we will add n to this number to get the desired solution.

As a last remark: in this part of the solution for brevity we will denote as palindromes only those that are nice by the definitions in the problem statement.

Code consisting of n digits 1 contains one n-digit palindrome, two $(n-1)$-digit palindromes, ..., $n-2$ three digit palindromes and $n-1$ two digit palindromes. After summing up we get that this code has $\frac{n(n-1)}{2}$ palindromes. Analogously the code consisting of n digits 0 contains the same number of palindromes.

We now consider the code which contains at least one digit one and at least one digit zero. Then each digit 1 except the rightmost one is the start of at least one palindrome (the sequence of digits starting with it and ending in the first digit one to the right of it is of the form $100\dots01$ and is thus a palindrome). Analogously we conclude that each digit 0 apart from the rightmost one is a start of at least one palindrome. As we have at least one digit 1 and one digit 0 we conclude that each code consists of at least $n-2$ palindromes (where we have deducted 2 for the rightmost digit 1 and 0).

By induction on n we will show that for each $n \in \mathbb{N}, n \geqslant 2$ we can find a code with exactly $n-2$ palindromes. We can note that for $n = 2, 3, 4$ this is possible as the examples are (10), $(101), (1101)$. Now let's assume that the induction claim holds for some $n \in \mathbb{N}, n \geqslant 4$, and let $(x_1 \dots x_n)$ be a code with exactly $n-2$ palindromes.

That code is certainly not $(011\dots1)$ or $(100\dots0)$ (similarly as in the case with all digits equal we conclude that these codes have $\frac{(n-1)(n-2)}{2} > n-2$ palindromes).

We now that each of the digits one/zero apart from the rightmost ones is the start of at least one palindrome. In order for total number of palindromes to be $n-2$ all such digits are starts of exactly one palindrome. As $(x_1 \dots x_n) \neq (011\dots1)$ and $(x_1 \dots x_n) \neq (100\dots0)$, digit x_1 is not the rightmost digit one/zero $\implies$ x_1 is the start of exactly one palindrome.

We now show that we can choose a digit x_0 such that $(x_0 x_1 x_2 \dots x_n)$ contains exactly $n-1$ palindromes. As there are $n-2$ palindromes $(x_1 x_2 \dots x_n)$ we need to show that we can choose x_0 such that x_0 is a start of exactly one palindrome in $(x_0 x_1 \dots x_n)$. We know that x_0 is a start of at least one palindrome so we actually only have to show it is a start of at most one palindromes.

Let's consider to which palindromes can x_0 be a start:

- $(x_0 x_1)$ is a palindrome $\iff x_0 = x_1$
- $(x_0 x_1 x_2)$ is a palindrome $\iff x_0 = x_2$
- $(x_0 x_1 x_2 \dots x_k x_{k+1})$ is a palindrome, for some $k \in \{2, 3, 4, \dots, n-1\} \iff x_0 = x_{k+1}$ and $(x_1 x_2 \dots x_k)$ is a palindrome

As there is exactly one palindrome for which x_1 is the start we conclude there is at most one palindrome such that x_0 is its start and it has the form as in the third case above. Thus there are at most three palindromes to which x_0 can be the first digit as we have two options for the choice of $x_0 \in \{0, 1\}$. Thus, by The Pigeonhole Principle we can choose a digit such that x_0 is a start of at most one palindrome, as desired.

Now using this and the remarks given before we have shown that the smallest possible number of nice palindromes with n digits is 1 (for $n = 1$) and $2n-2$ (for $n \geqslant 2$).

Problem 4. Given a triangle ABC let D, E, F be orthogonal projections from A, B, C to the opposite sides respectively. Let X, Y, Z denote midpoints of AD, BE, CF respectively. Prove that perpendiculars from D to YZ, from E to XZ and from F to XY are concurrent.

(Matija Bucić)

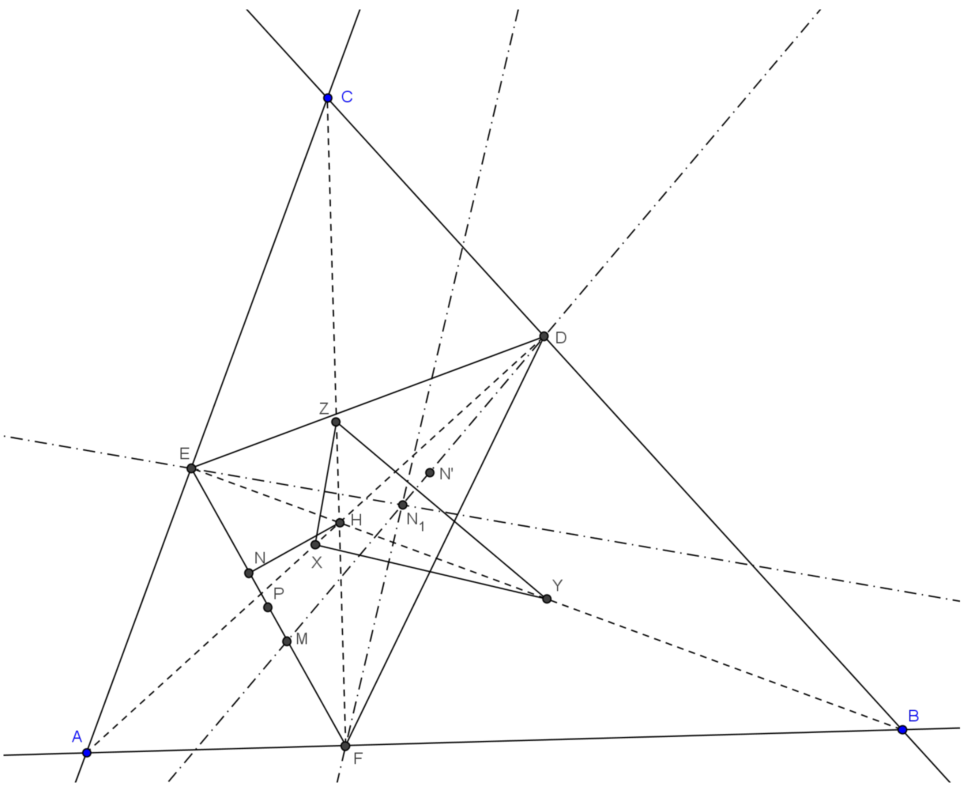
First solution. Let H be the orthocenter of the triangle ABC. We denote the midpoint of EF as P. As PZ is a midline of the triangle CEF we have $PZ||AC$, and as YH is perpendicular to AC, we get that YH is perpendicular to PZ. Analogously we conclude that the line ZH is perpendicular to PY, so H has to be the orthocenter of the triangle PYZ. From this we can deduce that the line PH is perpendicular to YZ, and thus PH is parallel to the line perpendicular to YZ which passes through D.

Now denote as N the tangency point of the incircle of the triangle DEF with its side EF. Let N' be the point symmetric to N with respect to H and let M be the tangency point of the D-excircle of the triangle DEF with the side EF. As P is the midpoint of $\overline{NM}$ and as is H the midpoint of NN', we have that PH is parallel to $N'M$. As we know that M is the map of the point N' under the homothety with centre D which maps the incircle to excircle of the triangle DEF, we can conclude that D, N' and M are collinear.

We can now conclude that the line perpendicular to YZ passing through D is parallel to PH while this line is parallel to $N'M$. As D lies on $N'M$ we conclude that DM is the line through D perpendicular to YZ.

Analogously we can conclude that perpendiculars from E to XZ and from F to XY are lines joining vertices with the corresponding excircle tangency point of the triangle DEF. Using the Ceva's Theorem gives us the result.

Remark: The intersection of the lines connecting the vertices of the triangle respective tangency points intersect in the point which is called *Nagel's* point of the triangle (so we have proved that the three lines in the problem intersect in the Nagel's point of the triangle DEF).



Second solution. By applying The Carnot's Theorem to the triangle XYZ and points D, E, F, three lines in the problem are concurrent if and only if:

$$FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = 0 \quad (1)$$

In the triangle AFD and EFB lines $\overline{FX}$ and $\overline{FY}$ are medians, so

$$FX^2 = \frac{1}{4}(2AF^2 + 2FD^2 - AD^2)$$
$$FY^2 = \frac{1}{4}(2FB^2 + FE^2 - EB^2).$$

Noting that the other sides on the LHS of (1) are medians in the respective triangles we deduce:

$$\begin{gathered}
FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = \\
\frac{1}{4}[(2AF^2 + \cancel{2FD^2} - \cancel{AD^2}) - (2FB^2 + \cancel{2FE^2} - \cancel{EB^2}) + \\
+(2DB^2 + \cancel{2DE^2} - \cancel{EB^2}) - (2DC^2 + \cancel{2DF^2} - \cancel{CF^2}) + \\
+(2EC^2 + \cancel{2EF^2} - \cancel{CF^2}) - (2EA^2 + \cancel{2ED^2} - \cancel{AD^2})] = \\
\frac{1}{2}(AF^2 - FB^2 + DB^2 - DC^2 + EC^2 - EA^2).
\end{gathered}$$

From right-angled triangles AFC and FBC we get:

$$AF^2 - FB^2 = (AC^2 - FC^2) - (BC^2 - FC^2) = AC^2 - BC^2.$$

Applying this analogously to triangles AEB, EBC, ADC, ADB we get:

$$\begin{gathered}
FX^2 - FY^2 + DY^2 - DZ^2 + EZ^2 - EX^2 = \\
\frac{1}{2}(AF^2 - FB^2 + DB^2 - DC^2 + EC^2 - EA^2) = \\
\frac{1}{2}(AC^2 - BC^2 + AB^2 - AC^2 + BC^2 - AB^2) = 0,
\end{gathered}$$

Q.E.D.