2ND European Mathematical Cup

7th December 2013–15th December 2013

Junior Category

Problem 1. For a positive integer m let $m?$ be the product of first m prime numbers.

Determine if there exist positive integers m and n with the following property:

$$m? = n(n+1)(n+2)(n+3).$$

(Matko Ljulj)

Problem 2. Let P be a point inside a triangle ABC. A line through P parallel to AB meets BC and CA at points L and F, respectively. A line through P parallel to BC meets CA and BA at points M and D respectively, and a line through P parallel to CA meets AB and BC at points N and E respectively. Prove

$$(PDBL) \cdot (PECM) \cdot (PFAN) = 8 \cdot (PFM) \cdot (PEL) \cdot (PDN),$$

where (XYZ) and $(XYZW)$ denote the area of the triangle XYZ and the area of quadrilateral $XYZW$.

(Steve Dinh)

Problem 3. We are given a combination lock consisting of 6 rotating discs. Each disc consists of digits $0, 1, 2, \ldots, 9$, in that order (after digit 9 comes 0). Lock is opened by exactly one combination. A move consists of turning one of the discs one digit in any direction and the lock opens instantly if the current combination is correct. Discs are initially put in the position 000000, and we know that this combination is not correct.

a) What is the least number of moves necessary to ensure that we have found the correct combination?

b) What is the least number of moves necessary to ensure that we have found the correct combination, if we know that none of the combinations 000000, 111111, 222222, …, 999999 is correct?

(Ognjen Stipetić, Grgur Valentić)

Problem 4. Let a, b, c be positive real numbers satisfying

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geqslant \frac{ab}{1+a+b} + \frac{bc}{1+b+c} + \frac{ca}{1+c+a}.$$

Prove

$$\frac{a^2+b^2+c^2}{ab+bc+ca} + a + b + c + 2 \geqslant 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}).$$

(Dimitar Trenevski)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.