1^{ST} European Mathematical Cup

24^{th} November 2012–1^{st} December 2012

Senior Category

Problem 1. Find all positive integers a, b, n and prime numbers p that satisfy

$$a^{2013} + b^{2013} = p^n.$$

(Matija Bucić)

Problem 2. Let ABC be an acute triangle with orthocenter H. Segments AH and CH intersect segments BC and AB in points A_1 and C_1 respectively. The segments BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D in AC. Prove that quadrilateral $APCD'$ is cyclic.

(Matko Ljulj)

Problem 3. Prove that the following inequality holds for all positive real numbers a, b, c, d, e and f:

$$\sqrt[3]{\frac{abc}{a+b+d}} + \sqrt[3]{\frac{def}{c+e+f}} < \sqrt[3]{(a+b+d)(c+e+f)}.$$

(Dimitar Trenevski)

Problem 4. Olja writes down n positive integers $a_1, a_2, \ldots, a_n$ smaller than p_n where p_n denotes the n-th prime number. Oleg can choose two (not necessarily different) numbers x and y and replace one of them with their product xy. If there are two equal numbers Oleg wins. Can Oleg guarantee a win?

(Matko Ljulj)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.