1^{ST} EUROPEAN MATHEMATICAL CUP
24^{th} November 2012–1^{st} December 2012
Senior Category

Problems and solutions

Problem 1. Find all positive integers a, b, n and prime numbers p that satisfy

$$a^{2013} + b^{2013} = p^n.$$

(Matija Bucić)

First solution. Let's denote $d = D(a, b), x = \frac{a}{d}, y = \frac{b}{d}$. With this we get

$$d^{2013}(a^{2013} + b^{2013}) = p^n.$$

So d must be a power of p, so let $d = p^k, k \in \mathbb{N}_0$. We can divide the equality by p^{2013k}. Now let's denote $m = n - 2013k, A = x^{671}, B = y^{671}$. So we get

$$A^3 + B^3 = p^m,$$

and after factorisation

$$(A + B)(A^2 - AB + B^2) = p^m.$$

(From the definition, A and B are coprime.)
Let's observe the case when some factor is 1: $A + B = 1$ is impossible as both A and B are positive integers. And $A^2 - AB + B^2 = 1 \Leftrightarrow (A - B)^2 + AB = 1 \Leftrightarrow A = B = 1$, so we get a solution $a = b = 2^k, n = 2013k + 1, p = 2, \forall k \in \mathbb{N}_0$.
If both factors are larger than 1 we have

$$p \mid A + B$$
$$p \mid A^2 - AB + B^2 = (A + B)^2 - 3AB$$
$$\implies p \mid 3AB.$$

If $p \mid AB$, in accordance with $p \mid A + B$ we get $p \mid A$ and $p \mid B$, which is in contradiction with A and B being coprime. So, $p \mid 3 \implies p = 3$.
Now we are left with 2 cases:

- First case: $A^2 - AB + B^2 = 3 \Leftrightarrow (A - B)^2 + AB = 3$ – so the only possible solutions are $A = 2, B = 1$ i $A = 1, B = 2$, but this turns out not to be a solution as $2 = x^{671}$ does not have a solution in positive integers.
- Second case: $3^2 \mid A^2 - AB + B^2$ – then we have:

$$3 \mid A + B \implies 3^2 \mid (A + B)^2$$
$$3^2 \mid A^2 - AB + B^2 = (A + B)^2 - 3AB$$
$$\implies 3^2 \mid 3AB$$
$$\implies 3 \mid AB.$$

And as we have already commented the case $p \nmid AB \implies$ doesn't have any solutions.

So all the solutions are given by

$$a = b = 2^k, n = 2013k + 1, p = 2, \forall k \in \mathbb{N}_0.$$

Second solution. As in the first solution, we take the highest common factor of a and b (which must be of the form p^k). Factorising the given equality we get

$$(x+y)(x^{2012}-x^{2011}y+x^{2010}y^2-\cdots-xy^{2011}+y^{2012})=p^m.$$

(We're using the same notation as in the first solution.) Denote the right hand side factor by A. As x and y are natural numbers, we have $x+y>1 \implies p \mid x+y$. So $p \nmid x$ and $p \nmid y$ (as x and y are coprime). Now by applying LTE (Lifting the Exponent Lemma):

$$\nu_p(x^{2013}+y^{2013})=\nu_p(x+y)+\nu_p(2013)$$

Now we know $\nu_p(2013)=0$ fo all primes p except $3, 11, 61$, and in the remaining cases $\nu_p(2013)=1$. Note $A=1$ and $(x,y)=(1,1)$ and $A>61$ for $(x,y)\neq(1,1)$. This inequality holds because for $(x,y)\neq(1,1)$ (WLOG $x \geqslant y$), we can write A as

$$x^{2011}(x-y)+x^{2009}y^2(x-y)+\cdots+xy^{2010}(x-y)+y^{2012},$$

which is greater than 61 in cases $x>y$ and $y\neq 1$.

- If $\nu_p(2013)=1 \implies \nu_p(A)=1 \implies A\in\{3,11,61\}$ which is clearly impossible.
- If $\nu_p(2013)=0 \implies \nu_p(A)=0 \implies A=1 \implies (x,y)=(1,1)$, so we get a solution

$$a=b=2^k, n=2013k+1, p=2, \forall k\in\mathbb{N}_0.$$

Problem 2. Let ABC be an acute triangle with orthocenter H. Segments AH and CH intersect segments BC and AB in points A_1 and C_1 respectively. The segments BH and A_1C_1 meet at point D. Let P be the midpoint of the segment BH. Let D' be the reflection of the point D in AC. Prove that quadrilateral $APCD'$ is cyclic.

(Matko Ljulj)

First solution. We shall prove that D is the orthocenter of triangle APC. From that the problem statement follows as

$$\begin{aligned}\angle AD'C=\angle ADC=180^\circ-\angle DAC-\angle DCA=(90^\circ-\angle DAC)+(90^\circ-\angle DCA)=\\ =\angle PCA+\angle PAC=180^\circ-\angle APC.\end{aligned}$$

We can note that quadrilateral BA_1HC_1 is cyclic. Lines BA_1 and C_1H intersect in C, lines BC_1 and A_1H intersect in A, lines BH and C_1A_1 intersect in D, and point P is the circumcenter of BA_1HC_1. So by the corollary of the Brocard's theorem point D is indeed the orthocenter of triangle APC as desired.

Second solution. Denote by B_1 the orthogonal projection of B on AC. By cyclic quadrilaterals $B_1C_1PA_1$ (Euler's circle), HA_1CB_1, AC_1A_1C and C_1HB_1A we get the following equations:

$$\angle A_1PB_1=\angle DC_1B_1$$
$$\angle A_1B_1P=\angle A_1CC_1=\angle A_1AC_1=\angle DB_1C_1.$$

From these equalities we get that triangles B_1PA_1 and B_1C_1D are similar, which implies

$$\frac{|B_1D|}{|B_1A_1|}=\frac{|B_1C_1|}{|B_1P|}\implies |B_1A_1|\cdot|B_1C_1|=|B_1D|\cdot|B_1P|.$$

Analogously, using cyclic quadrilateral ABA_1B_1 and C_1BCB_1 we get the following angle equations:

$$\angle B_1AC_1=180^\circ-\angle B_1A_1B=\angle B_1A_1C$$
$$\angle AB_1C_1=180^\circ-\angle C_1B_1C=\angle CBA=180^\circ-\angle A_1B_1A=\angle A_1B_1C.$$

From these equalities we get that triangles B_1AC_1 and B_1A_C are similar so

$$\frac{|B_1C_1|}{|B_1C|}=\frac{|AB_1|}{|A_1B_1|}\implies |B_1A_1|\cdot|B_1C_1|=|B_1A|\cdot|B_1C|.$$

Thus we get $|B_1D'|\cdot|B_1P|=|B_1D|\cdot|B_1P|=|B_1A_1|\cdot|B_1C_1|=|B_1A|\cdot|B_1C|$ so by the reverse of the power of the point theorem the quadrilateral $APCD'$ is cyclic as desired.

Problem 3. Prove that the following inequality holds for all positive real numbers a, b, c, d, e and f:

$$\sqrt[3]{\frac{abc}{a+b+d}}+\sqrt[3]{\frac{def}{c+e+f}}<\sqrt[3]{(a+b+d)(c+e+f)}.$$

(Dimitar Trenevski)

Solution. The inequality is equivalent to

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}+\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}<1.$$

By AM-GM inequality we have

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}\leqslant\frac{1}{3}\left(\frac{a}{a+b+d}+\frac{b}{a+b+d}+\frac{c}{c+e+f}\right),$$

$$\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}\leqslant\frac{1}{3}\left(\frac{d}{a+b+d}+\frac{e}{c+e+f}+\frac{f}{c+e+f}\right).$$

Adding the inequalities we get

$$\sqrt[3]{\frac{abc}{(a+b+d)^2(c+e+f)}}+\sqrt[3]{\frac{def}{(a+b+d)(c+e+f)^2}}\leqslant\frac{1}{3}\left(\frac{a+b+d}{a+b+d}+\frac{c+e+f}{c+e+f}\right)=\frac{2}{3}<1,$$

as desired.

Problem 4. Olja writes down n positive integers $a_1,a_2,\dots,a_n$ smaller than p_n where p_n denotes the n-th prime number. Oleg can choose two (not necessarily different) numbers x and y and replace one of them with their product xy. If there are two equal numbers Oleg wins. Can Oleg guarantee a win?

(Matko Ljulj)

Solution. For $n=1$, Oleg won't be able to write 2 equal numbers on the board as there will be only one number written on the board. We shall now consider the case $n>2$.

Let's note that as all the numbers are strictly smaller than p_n we have all their prime factors are from the set $\{p_1,p_2,\dots,p_{n-1}\}$, so there are at most $n-1$ of them in total. We will represent each number $a_1,a_2,\dots,a_n$ by the ordered $(n-1)$-tuple of non-negative integers in the following way if $a_i=p_1^{\alpha_{i,1}}\cdot p_2^{\alpha_{i,2}}\cdot\dots\cdot p_{n-1}^{\alpha_{i,(n-1)}}$, then we assign $v_i=(\alpha_{i,1},\alpha_{i,2},\dots,\alpha_{i,(n-1)})$, for all $i\in\{1,2,\dots,n\}$.

Let's consider the following system of equations:

$$\begin{gathered}\alpha_{1,1}x_1+\alpha_{2,1}x_2+\dots+\alpha_{n,1}x_n=0\\ \alpha_{1,2}x_1+\alpha_{2,2}x_2+\dots+\alpha_{n,2}x_n=0\\ \dots\\ \alpha_{1,(n-1)}x_1+\alpha_{2,(n-1)}x_2+\dots+\alpha_{n,(n-1)}x_n=0\end{gathered}$$

There is a trivial solution $x_1=x_2=\dots=x_n=0$. But as this system has less equalities than variables we can deduce that it has infinitely many solutions in the set of rational numbers (as all the coefficients are rational). Let $(y_1,y_2,\dots,y_n)$ be a not trivial solution (so the solution in which not all of y_i equal 0). Then we can rewrite the initial system using $a_1,a_2,\dots,a_n$:

$$\prod_{i=1}^{n}a_i^{y_i}=\prod_{i=1}^{n}p_1^{\alpha_{i,1}y_i}\cdot p_2^{\alpha_{i,2}y_i}\cdot\dots\cdot p_{n-1}^{\alpha_{i,(n-1)}y_i}=\prod_{j=1}^{n-1}p_j^{\alpha_{1,j}y_1+\alpha_{2,j}y_2+\dots+\alpha_{n,j}y_n}=\prod_{j=1}^{n-1}p_j^0=1$$

$$\implies\prod_{i=1}^{n}a_i^{y_i}=1.$$

Considering the numbers $y_1,y_2,\dots,y_n$ as rational numbers in which the respective nominator and denominator are coprime, Denote by L the lowest common multiplier of their denominators. Taking the L-th power of the upper equality we get integer exponents in the upper equation (which don't have a common factor). Furthermore, WLOG we can assume that $a_1,a_2,\dots,a_k$ are those elements a_i whose exponents are negative and numbers $a_{k+1},a_{k+2},\dots,a_{k+l}$ are those elements with postivie exponent (for some $k, l \in \mathbb{N}$, $k+l \leqslant n$). Then, when we shift all a_i-s with negative exponent to the opposite side of the equation and when those with zero exponent get ruled out we get that the following equality

$$\prod_{i=1}^{k} a_i^{r_i} = \prod_{i=k+1}^{l} a_i^{r_i} \tag{1}$$

holds for some positive integers $r_1, r_2, \ldots, r_{k+l}$ for which $D(r_1, r_2, \ldots, r_{k+l}) = 1$ and for some numbers $a_1, a_2, \ldots, a_{k+l}$. (We can note that there is at least one number a_i on both sides of the equality otherwise we have only ones on the board.)

We shall prove that there is a sequence of transformations by which using this relation we will get two equal numbers among $a_1, a_2, \ldots a_n$.

Lemma 1. *Let $(a, b) \in \mathbb{N}^2$ and $(x_1, x_2) \in \mathbb{N}^2$ be such that $GCD(x_1, x_2) = 1$. Then there exists a sequence of transformations which replaces the numbers (a, b) with (a', b'), where one of these numbers a', b' is equal to $a^{x_1} b^{x_2}$.*

Proof. We'll prove this by induction on $x_1 + x_2$, for all $(a, b) \in \mathbb{N}^2$. As the basis consider $x_1 + x_2 = 2 \implies x_1 = x_2 = 1$. The number ab we can get by applying transformation $(a, b) \to (a, ab)$.

Let's assume that the claim holds for all (x_1, x_2) such that $x_1 + x_2 < n$, and for all (a, b). Let's take some numbers (x_1, x_2) such that $x_1 + x_2 = n$ and some arbitrary numbers (a, b). If $x_1 = x_2$ is satisfied, since x_1 and x_2 are coprime, we could conclude that both numbers are equal to 1, but we have already proved this case in basis. Let's assume $x_1 \neq x_2$. WLOG $x_1 > x_2$. Then we apply the transformation $(a, b) \to (a, ab)$, and then apply the induction hypothesis on numbers (a, ab) and $(x_1 - x_2, x_2)$:

$$(a, b) \to (a, ab) \to (\gamma, a^{x_1 - x_2} (ab)^{x_2}) = (\gamma, a^{x_1} b^{x_2}),$$

where γ is some positive integer, what we wanted to prove. □

Lemma 2. *Let $k \in \mathbb{N}$, $(b_1, b_2, \ldots b_k) \in \mathbb{N}^k$ and $(x_1, x_2, \ldots x_k) \in \mathbb{N}^k$. Then there exists sequence of transformations which instead of numbers $(b_1, b_2, \ldots b_k)$ writes down numbers $(b_1', b_2', \ldots b_k')$ such that one of those numbers is equal to*

$$(b_1^{x_1} b_2^{x_2} \cdots b_k^{x_k})^{\frac{1}{d}},$$

where d denotes greatest common divisor of numbers $x_1, x_2, \ldots x_k$.

Proof. Intuitively, this lemma is just *Lemma 1* repeated $(k-1)$ times.

We'll prove this by induction on k, for all $b_1, b_2, \ldots b_k$ and $x_1, x_2, \ldots x_k$. In the basis, for $k = 1$, it holds $d = x_1$, so it we don't have to do any transformation to reach desired situation.

Let's assume that the claim holds for some $k \in \mathbb{N}$. Let's take arbitrary $(b_1, b_2, \ldots b_k, b_{k+1})$ and $(x_1, x_2, \ldots x_k, x_{k+1})$. Then we apply *Lemma 1* on numbers (b_k, b_{k+1}) and (x_k', x_{k+1}'), where $x_k' = \frac{x_k}{d_1}$, $x_{k+1}' = \frac{x_{k+1}}{d_1}$, $d_1 = GCD(x_k, x_{k+1})$, and then we apply the induction hypothesis on numbers $(b_1, b_2, \ldots b_k^{x_k'} b_{k+1}^{x_{k+1}'})$ and $(x_1, x_2, \ldots x_{k-1}, d_1)$:

$$(b_1, b_2, \ldots b_k, b_{k+1}) \to (b_1, b_2, \ldots b_{k-1}, \gamma_k, b_k^{x_k'} b_{k+1}^{x_{k+1}'}) \to (\gamma_1, \gamma_2, \ldots, \gamma_k, (b_1^{x_1} b_2^{x_2} \cdots b_{k-1}^{x_{k-1}} (b_k^{x_k'} b_{k+1}^{x_{k+1}'})^{d_1})^{\frac{1}{d_2}}),$$

where $\gamma_1, \gamma_2, \ldots, \gamma_k$ are some positive integers and $d_2 = GCD(x_1, x_2, \ldots x_{k-1}, d) = GCD(x_1, x_2, \ldots x_{k-1}, x_k, x_{k+1}) = d$. Notice that last number in upper relation is the one we wanted to get. □

Lemma 3. *Let $(a, b) \in \mathbb{N}^2$ and $(x_1, x_2) \in \mathbb{N}^2$ such that $GCD(x_1, x_2) = 1$. Then there exists sequence of transformations which instead of numbers (a, b) writes down numbers (a', b') for which it is satisfied $a'/b' = a^{x_1}/b^{x_2}$.*

Proof. We'll prove this by induction on $x_1 + x_2$, for all $(a, b) \in \mathbb{N}^2$. In the basis is $x_1 + x_2 = 2 \implies x_1 = x_2 = 1$, so we don't have to do any transformation to reach desired situation.

Ler's assume that the claim hold for all (x_1, x_2) such that $x_1 + x_2 < n$, and for all (a, b). Let's take some numbers (x_1, x_2) such that $x_1 + x_2 = n$ and arbitrary numbers (a, b).

- If one of the numbers x_1 and x_2 is even (WLOG x_1 is even): we apply tranformation $(a, b) \to (a^2, b)$, and then we apply induction hypothesis on numbers (a^2, b) and $(\frac{x_1}{2}, x_2)$.
- Both numbers x_1 and x_2 are odd, and they are equal: then they are both equal to 1, which we have already solved in the basis.
- Numbers x_1 and x_2 are odd and distinct (WLOG $x_1 > x_2$): we make following transformations $(a, b) \to (a, ab) \to (a^2, ab)$, and then we apply induction hypothesis on numbers (a^2, ab) and $(\frac{x_1+x_2}{2}, x_2)$:

$$(a, b) \to (a, ab) \to (a^2, ab) \to (c \cdot (a^2)^{\frac{x_1+x_2}{2}}, c \cdot (ab)^{x_2}) = ((a^{x_2} c) \cdot a^{x_1}, (a^{x_2} c) \cdot b^{x_2}),$$

where c is some positive integer, what we wanted to prove. □

In the equality (1), let $d_1 = GCD(r_1, r_2, \dots, r_k)$, $d_2 = GCD(r_{k+1}, r_{k+2}, \dots, r_{k+l})$, $z_i = \frac{r_i}{d_1}$, $\forall i \in \{1, 2, \dots, k\}$, $z_i = \frac{r_i}{d_2}$, $\forall i \in \{k+1, k+2, \dots, k+l\}$. As well let A be the left hand side of the equality (1), and let B be the right hand side. Let $A' = A^{\frac{1}{d_1}}$ and $B' = B^{\frac{1}{d_2}}$. We want to do such transformations that we get x i y which will have same ratio as A and B.

If we apply *Lemma 2* on the numbers $(a_1, a_2, \dots, a_k)$ and $(z_1, z_2, \dots, z_k)$; we get (among other numbers we get) the number A'. As well applying the same lemma on the numbers $(a_{k+1}, a_{k+2}, \dots, a_{k+l})$ and $(z_{k+1}, z_{k+2}, \dots, z_{k+l})$, we will get the number B' on the board.

Numbers d_1 and d_2 are coprime (otherwise there would be some prime p which would divide d_1 and d_2 which would imply it divides $r_1, r_2, \dots, r_{k+l}$ as well which is in contradiction to the assumption they do not have a common factor). So we can apply *Lemma 3* on the numbers (A', B') and (d_1, d_2). Now we get two numbers with the same ratio as A i B. But as by (1) we have $A = B$, we get 2 equal numbers on the board.

Thus Oleg can guarantee a win for any $n > 1$.

Comment: We can get to the relation (1) by concluding that the set $\{v_1, v_2, \dots, v_n\}$ is linearly dependant subset of $(n-1)$-dimensional space $\mathbb{Q}^{n-1}$.