MLADI NADARENI MATEMATIČARI
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1st BALKAN STUDENT MATHEMATICAL COMPETITION

1. Matematičko natjecanje učenika Balkana

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3rd and 4th grade

Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for every two real numbers x and y,

$$f\left(f\left(x\right)+xy\right)=f\left(x\right)\cdot f\left(y+1\right).$$

(Marko Radovanović)

Problem 2. Paralampius the Gnu stands on number 1 on number line. He wants to come to a natural number k by a sequence of consecutive jumps. Let us denote the number of ways on which Paralampius can come from number 1 to number k with $f\left(k\right)$ ($f\colon \mathbb{N} \to \mathbb{N}_0$). Specially, $f\left(1\right)=0$. A way is a sequence of numbers (with order) which Paralampius has visited on his travel from number 1 to number k. Paralampius can, from number b, jump to number

- $2b$ (always),
- $3b$ (always),
- b^2 $\left(\text{if } \dfrac{b^4}{6k} \in \mathbb{N}, \text{ where } k \text{ is a natural number on which he wants to come to in the end}\right)$.

Prove that, for every natural number n, there exists a natural number m_0 such that for every natural number $m > m_0$,

$$f\left(m\right) < 2^{\alpha_1+\alpha_2+\ldots+\alpha_i-n},$$

where $m = p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\ldots\cdot p_i^{\alpha_i}$ ($p_1 < p_2 < \ldots < p_i$ are prime divisors of number m and i, α_1, α_2, $\ldots$, α_i are natural numbers) is a prime factorization of natural number m. It is known that this factorization is unique for every natural number $m > 1$.

(Melkior Ornik, Ivan Krijan)

Problem 3. A convex n-gon ($n \in \mathbb{N}$, $n > 2$) is given in the plane. Its area is less than 1. For each point X of this plane, we shall denote with $F\left(X\right)$ the area of the convex hull of point X and a given n-gon (the area of the minimal convex polygon which includes both the point X and a given n-gon). Prove that the set of points for which $F\left(X\right)=1$ is a convex polygon with $2n$ sides or less.

Problem 4. Prove that for every natural number k, there exists infinitely many natural numbers n such that

$$\frac{n-d\left(n^r\right)}{r} \in \mathbb{Z}, \text{ for every } r \in \left\{1,\, 2,\, \ldots,\, k\right\}.$$

Here, $d\left(x\right)$ denotes the number of natural divisors of a natural number x, including 1 and x itself.

(Melkior Ornik)

Time allowed: 240 minutes.
Each problem is worth 10 points.
Write each problem on a separate paper.
Calculators or any other helping items, excluding rulers and compasses, are not allowed.