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1st BALKAN STUDENT MATHEMATICAL COMPETITION

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Solutions

Problem 1. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for every two real numbers x and y,

$$f\left(f\left(x\right)+xy\right)=f\left(x\right)\cdot f\left(y+1\right).$$

(Marko Radovanović)

Solution. Obvious solutions are $f(x) \equiv 0$, $\forall x \in \mathbb{R}$; $f(x) \equiv 1$, $\forall x \in \mathbb{R}$ and $f(x) \equiv x$, $\forall x \in \mathbb{R}$.
(1 point)

Let $x := 0$, $y := t-1$. Then, $f(f(0)) = f(0)\, f(t)$, $\forall t \in \mathbb{R}$. (1 point)

Let us consider two cases.

1° $f(0) \neq 0$. Then, for each $t \in \mathbb{R}$, it is true that $f(t) = \dfrac{f(f(0))}{f(0)} = k$. (1 point)

By plugging $t = 0$, we reach $f(0)^2 = f(f(0))$ or $k^2 = k$. If $k = 0$, we get $f(0) = 0$, which gives a contradiction. So, $k = 1$ must hold. In that case, we reach this solution: $f(x) \equiv 1$, $\forall x \in \mathbb{R}$.
(2 points)

2° $f(0) = 0$. Now we shall consider following two cases:

- $f(x) = 0$ if and only if $x = 0$. Then, if we plug $y := -1$, we reach

$$f\left(f\left(x\right)-x\right)=f\left(x\right)f\left(0\right)=0 \Longrightarrow f\left(x\right)-x=0 \Longrightarrow f\left(x\right)=x.$$

  So, in this case we reach this solution: $f(x) \equiv x$, $\forall x \in \mathbb{R}$. (2 points)

- There exists a real number $x_0 \neq 0$ such that $f(x_0) = 0$. Then, let $x := x_0$ and $y := \dfrac{t}{x_0}$ for some real number t. Then,

$$f\left(f\left(x_0\right)+x_0\cdot\frac{t}{x_0}\right)=f\left(x_0\right)f\left(\frac{t}{x_0}+1\right)=0 \Longrightarrow f\left(t\right)=0.$$

  Finally, we reach the solution $f(x) \equiv 0$, $\forall x \in \mathbb{R}$. (2 points)

We can see that functions mentioned in the beginning (and only those functions) satisfy the conditions of the problem.
(1 point)

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Problem 2. Paralampius the Gnu stands on number 1 on number line. He wants to come to a natural number k by a sequence of consecutive jumps. Let us denote the number of ways on which Paralampius can come from number 1 to number k with $f(k)$ ($f\colon \mathbb{N} \to \mathbb{N}_0$). Specially, $f(1) = 0$. A way is a sequence of numbers (with order) which Paralampius has visited on his travel from number 1 to number k. Paralampius can, from number b, jump to number

- $2b$ (always),
- $3b$ (always),
- b^2 $\left(\text{if } \dfrac{b^4}{6k} \in \mathbb{N}, \text{ where } k \text{ is a natural number on which he wants to come to in the end}\right)$.

Prove that, for every natural number n, there exists a natural number m_0 such that for every natural number $m > m_0$,

$$f(m) < 2^{\alpha_1+\alpha_2+\ldots+\alpha_i-n},$$

where $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_i^{\alpha_i}$ ($p_1 < p_2 < \ldots < p_i$ are prime divisors of number m and i, α_1, α_2, …, α_i are natural numbers) is a prime factorization of natural number m. It is known that this factorization is unique for every natural number $m > 1$.

(Melkior Ornik, Ivan Krijan)

Solution. Let's notice that Paralampius can only reach numbers of the form $2^a \cdot 3^b$ $(a, b \in \mathbb{N}_0)$. That is because neither one of the allowed jumps "introduces" a prime factor different than 2 or 3 (first jump "introduces" 2, second one "introduces" 3, while the last one doubles the number of each of already existing prime numbers). Therefore, for each natural number m divisible by a prime number $p > 3$, it holds that $f(m) = 0 < 2^{\alpha_1+\alpha_2+\ldots+\alpha_i-n}$, $\forall n \in \mathbb{N}$. Specially, the same claim holds for $f(1)$. Let us continue by observing only numbers of the form $2^a \cdot 3^b$ $(a, b \in \mathbb{N}_0)$, where at least one of numbers a and b differs from 0. Let's introduce the following symbols.

1. (k, l), $k, l \in \mathbb{N}_0$, $k^2 + l^2 \geq 1$ will denote number $2^k \cdot 3^l$. This k can be any nonnegative integer – it has no connection to k from the text of the problem.
2. $g\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$, $g(k, l) = f\left(2^k \cdot 3^l\right)$, $k^2 + l^2 \geq 1$.

Now we can write that Paralampius can jump from pair (k, l) to pair

- $(k+1, l)$ (always),
- $(k, l+1)$ (always),
- $(2k, 2l)$ (if $4k \geq a+1$ and $4l \geq b+1$ with pair (a, b) being the goal).

Let pair (a, b) in further text always mark the number which is Paralampius' goal. Let's show that Paralampius can jump according to the third rule once, at most. Let's assume the opposite, that is, that Paralampius has jump from pair (k, l) to pair $(2k, 2l)$ and after a couple of jumps, again from pair (k_1, l_1) to pair $(2k_1, 2l_1)$. It is obvious that none of these rules decrease any element in a pair. On the contrary, every rule increases at leas one number in a pair, while the third rule increases (doubles) both numbers. So, it is true that $k_1 \geq 2k \geq \dfrac{a+1}{2}$, so $2k_1 \geq a+1 > a$, and, as it has been mentioned previously, none of the rules will decrease numbers in a pair, so Paralampius will never be able to reach pair (a, b). Contradiction! So, Paralampius will jump according to the third rule at mose once. If he never jumps according to the third rule, he can jump from $(0, 0)$ to (a, b) in

$$\binom{a+b}{b}$$

ways. Further, if Paralampius jumps once according to the third rule, for instance from pair (k, l) to pair $(2k, 2l)$, then it must hold for k and l that $4k \geq a+1$ and $4l \geq b+1$ (condition of the problem), but also (to keep Paralampius from "overjumping" the goal, because, as we have shown, he can not return) $2k \leq a$ and $2l \leq b$. So, Paralampius can come from $(0, 0)$ to (a, b), if he jumps once from (k, l) to $(2k, 2l)$ in

$$\binom{k+l}{l} \cdot \binom{a+b-2k-2l}{b-2l}$$

ways. According to that, Paralampius can come from (0, 0) to $(a,\, b)$ in, altogether,

$$g(a,\, b) = \binom{a+b}{b} + \sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} \binom{k+l}{l} \cdot \binom{a+b-2k-2l}{b-2l}$$

ways. (2 points)

Now our problem boils down to showing that for each natural number n there exists a natural number m_0 such that for every natural number $2^a \cdot 3^b > m_0$ $\left(a,\, b \in \mathbb{N}_0,\, a^2 + b^2 \geq 1\right)$, it holds that

$$g(a,\, b) < 2^{a+b-n}.$$

Let us now show three lemmas.

Lemma 1. For all natural numbers x and y, $\binom{x}{y} \leq 2^{x-1}$ holds.

Proof. We know that $\binom{x}{y} \leq \binom{x}{\lfloor \frac{x}{2} \rfloor}$, so it suffices to prove that, for every natural number x, $\binom{x}{\lfloor \frac{x}{2} \rfloor} \leq 2^{x-1}$. We will show this using mathematical induction. For $x = 1$, $\binom{1}{0} \leq 2^{1-1}$. Let's assume that for some $x \in \mathbb{N}$, $\binom{x}{\lfloor \frac{x}{2} \rfloor} \leq 2^{x-1}$. Then,

$$\binom{x+1}{\lfloor \frac{x+1}{2} \rfloor} = \binom{x}{\lfloor \frac{x+1}{2} \rfloor} + \binom{x}{\lfloor \frac{x-1}{2} \rfloor} \leq 2^{x-1} + 2^{x-1} = 2^x.$$

We shall notice that equality is only possible if $x = 1$ or $x = 2$. This proves our first lemma. (1 point) ∎

Lemma 2. There exists a natural number c_1 such that, if $a > c_1$ or $b > c_1$ (then $a + b > c_1$), then

$$(a+3)(b+3) < 2^{\frac{a+b}{4}}.$$

Proof. If we fix $a + b$, then the left side reaches its maximum for $a = b$. Let then be $M = a + b + 6$. Then,

$$M = (a+3) + (b+3) \geq \text{ [using AM-GM inequality] } \geq 2\sqrt{(a+b)(b+3)}.$$

So, it suffices to show that there exists a natural number c_1 such that for every natural number $x > c_1$ it holds that

$$(x+3)^2 < 2^{\frac{x}{2}} \iff (x+3)^4 < 2^x.$$

It is obvious that the left side of this inequality "grows more slowly" than the right side, so it is enough to show that there exists at least one such natural number x. We can directly see that, for example, $x = 20$ has this property. (1 point) ∎

Lemma 3. For every natural number n there exists natural number c such that, if $a > c$ or $b > c$ (then $a + b > c$), then

$$\binom{a+b}{b} < 2^{a+b-n-1}.$$

Proof. We know that

$$\binom{a+b}{b} \leq \binom{a+b}{\lfloor \frac{a+b}{2} \rfloor},$$

so, in general,

$$\binom{a+b}{b} \le \binom{2x}{x}, \quad \text{where} \quad x = \left\lceil \frac{a+b}{2} \right\rceil.$$

Now, it is enough to show that for each natural number n there exists natural number c such that, if $x > c$, then

$$\binom{2x}{x} < 2^{2x-n-1}.$$

Further, let us notice that $\binom{2x+2}{x+1} < 4 \cdot \binom{2x}{x} \iff 2 > 0$, and $2^{2x+2-n-1} = 4 \cdot 2^{2x-n-1}$. So, it inductively follows that, if we show that there exists $c \in \mathbb{N}$ such that $\binom{2c}{c} < 2^{2c-n-1}$, then, for every natural number $x > c$, $\binom{2x}{x} < 2^{2x-n-1}$ will hold. Let us now prove that such $c \in \mathbb{N}$ exists.

$$\begin{aligned}
& \binom{2c}{c} < 2^{2c-n-1} \\
\iff & \frac{(2c)!}{c! \cdot c!} < 2^{2c-n-1} \\
\iff & \frac{(2c)(2c-1)(2c-2)\cdot \ldots \cdot 2 \cdot 1}{c^2 (c-1)^2 (c-2)^2 \cdot \ldots \cdot 2^2 \cdot 1^2} < 2^{2c-n-1} \\
\iff & \frac{2^c \cdot c(c-1)(c-2)\cdot \ldots \cdot 2 \cdot 1 \cdot (2c-1)(2c-3)\cdot \ldots \cdot 3 \cdot 1}{c^2 (c-1)^2 \cdot \ldots \cdot 2^2 \cdot 1^2} < 2^{2c-n-1} \\
\iff & \frac{(2c-1)(2c-3)\cdot \ldots \cdot 3 \cdot 1}{c(c-1)\cdot \ldots \cdot 2 \cdot 1} < 2^{c-n-1} \\
\iff & \frac{(2c-1)(2c-3)\cdot \ldots \cdot 3 \cdot 1}{2^c \cdot c(c-1)\cdot \ldots \cdot 2 \cdot 1} < 2^{-n-1} \\
\iff & \frac{(2c)(2c-2)(2c-4)\cdot \ldots \cdot 4 \cdot 2}{(2c-1)(2c-3)\cdot \ldots \cdot 3 \cdot 1} > 2^{n+1} \\
\iff & \left(1 + \frac{1}{2c-1}\right)\left(1 + \frac{1}{2c-3}\right) \cdot \ldots \cdot \left(1 + \frac{1}{1}\right) > 2^{n+1}.
\end{aligned}$$

Obviously $\left(1 + \frac{1}{2c-1}\right)\left(1 + \frac{1}{2c-3}\right) \cdot \ldots \cdot \left(1 + \frac{1}{1}\right) \ge \frac{1}{2c-1} + \frac{1}{2c-3} + \ldots + \frac{1}{1}$ for every natural number c, so it suffices to show that there exists a natural number c such that

$$\frac{1}{2c-1} + \frac{1}{2c-3} + \ldots + \frac{1}{1} > 2^{n+1}.$$

Since $\frac{1}{2c-1} + \frac{1}{2c-3} + \ldots + \frac{1}{1} \ge \frac{1}{2c} + \frac{1}{2c-2} + \ldots + \frac{1}{2}$, that is,

$$\frac{1}{2c-1} + \frac{1}{2c-3} + \ldots + \frac{1}{1} \ge \frac{\frac{1}{c} + \frac{1}{c-1} + \ldots + \frac{1}{1}}{2},$$

it is enough for us to show that there exists a natural number c such that

$$\frac{1}{c} + \frac{1}{c-1} + \ldots + \frac{1}{1} > 2^{n+2}.$$

Let's notice that for every nonnegative integer t, $\frac{1}{2^t+1} + \frac{1}{2^t+2} + \ldots + \frac{1}{2^{t+1}} \ge \frac{1}{2}$. This follows from the fact that, on the left side of the inequality, we have 2^t summands, each one of which is larger or equal than $\frac{1}{2^{t+1}}$. Now it directly follows that

$$\frac{1}{2^{2^{n+3}}} + \frac{1}{2^{2^{n+3}}-1} + \frac{1}{2^{2^{n+3}}-2} + \ldots + \frac{1}{1} > 2^{n+2}.$$

With this, we have finally proven this lemma. (3 points)

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Now, we wish to show that for every natural number n there exists a natural number t such that, if $a > t$ or $b > t$, then

$$\sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} \binom{k+l}{l} \cdot \binom{a+b-2k-2l}{b-2l} < 2^{a+b-n-1}.$$

Further, let's fix n. Let t be such natural number that $t > c_1$ and $t > c$, where c_1 and c are numbers of **Lemma 2**, that is, from **Lemma 3**. Let us observe numbers defined by pair $(a,\, b)$, where $a+1 > 4t$ or $b+1 > 4t$. Then, if Paralampius jumps on his travel from pair $(k,\, l)$ to pair $(2k,\, 2l)$, for numbers k and l it holds that $k+l > t$ because $k \geq \dfrac{a+1}{4}$ and $l \geq \dfrac{b+1}{4}$, so $k > t$ or $l > t$. So, because of **Lemma 3**, $\binom{k+l}{l} < 2^{k+l-n-1}$. According to **Lemma 1**, we have that for all $x,\, y \in \mathbb{N}$, it holds that $\binom{x}{y} \leq 2^{x-1}$, so, because of that,

$$\sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} \binom{k+l}{l} \cdot \binom{a+b-2k-2l}{b-2l} < \sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} 2^{k+l-n-1} \cdot 2^{a+b-2k-2l-1}.$$

Since $k > \dfrac{a}{4}$ and $l > \dfrac{b}{4}$, we can see that it suffices to show that (we set k to be exactly equal to $\dfrac{a}{4}$ and the same for l)

$$\sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} 2^{\frac{3}{4}a+\frac{3}{4}b-n-2} < 2^{a+b-n-1}.$$

It is obvious that there exists at most $\dfrac{a}{2} - \dfrac{a+1}{4} + 1 = \dfrac{a+3}{4}$ possible numbers k. The same is true for numbers l. So, it is enough for us to show

$$(a+3)\,(b+3) \cdot 2^{\frac{3}{4}a+\frac{3}{4}b-n-5} < 2^{a+b-n}.$$

Since $a > 4t-1$ or $b > 4t-1$, then $a > t$ or $b > t$ and, since $t > c_1$, where c_1 is the number from **Lemma 2**, it is obvious that $a > c_1$ or $b > c_1$, so, because of **Lemma 2**, wanted inequality directly follows. With this we have shown that for every natural number n, there exists a natural number t such that, if $a > t$ or $b > t$, then

$$\sum_{\frac{a+1}{4} \leq k \leq \frac{a}{2}} \sum_{\frac{b+1}{4} \leq l \leq \frac{b}{2}} \binom{k+l}{l} \cdot \binom{a+b-2k-2l}{b-2l} < 2^{a+b-n-1}.$$

In our case, this natural number t is such that $t > 4c_1$ and $t > 4c$, where c_1 and c are numbers from **Lemma 2**, that is, from **Lemma 3**. (2 points)

It is obvious that, for such natural number t, because of **Lemma 3**, the following holds:

$$\binom{a+b}{b} < 2^{a+b-n-1}.$$

By adding the last two inequalities (we have shown that they hold), we reach that

$$g\,(a,\, b) < 2^{a+b-n},$$

what we have wanted to show. That is, we have shown that for every natural number n there exists a natural number m_0 such that for every natural number $2^a \cdot 3^b > m_0$ $\left(a,\, b \in \mathbb{N}_0,\, a^2+b^2 \geq 1\right)$, it holds that

$$g\,(a,\, b) < 2^{a+b-n}.$$

(1 point)

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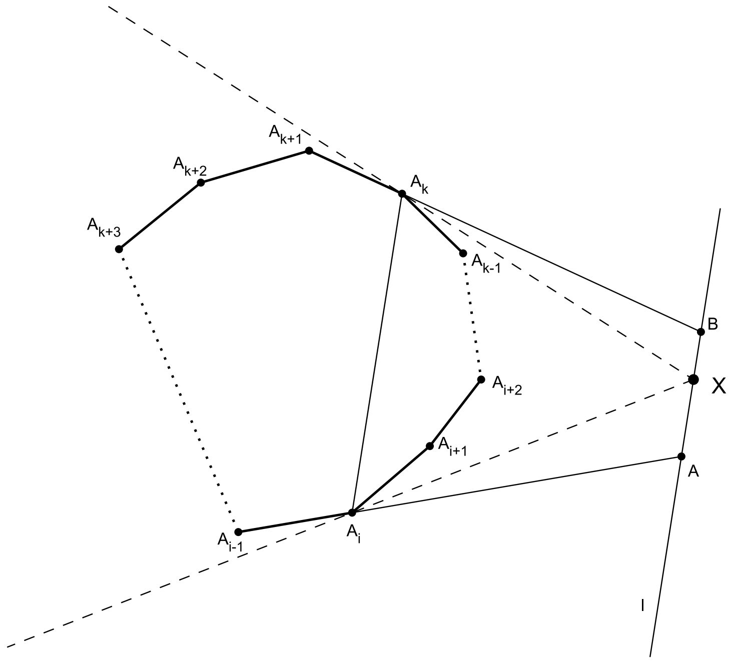
Problem 3. A convex n-gon ($n \in \mathbb{N}$, $n > 2$) is given in the plane. Its area is less than 1. For each point X of this plane, we shall denote with $F(X)$ the area of the convex hull of point X and a given n-gon (the area of the minimal convex polygon which includes both the point X and a given n-gon). Prove that the set of points for which $F(X) = 1$ is a convex polygon with $2n$ sides or less.

Solution. Let S be the set of all points T of the given plane such that $F(T) = 1$. If is obvious that all these points T are outside the given n-gon. Area of the given n-gon is less than 1, and when point T belongs to the interior (or the edge) of the given n-gon, it itself becomes our wanted convex hull. (1 point)
Point T is outside the n-gon if there exists a line which passes through T such that it has no common points with the n-gon. Let's observe the following image.



Let $A_0, A_1, \ldots, A_{n-1}$ be vertices of the given n-gon. Also, let $A_j = A_{(j \bmod n)}$ $(j \in \mathbb{Z})$, where $(j \bmod n) = x$ is a number from the set $\{0, 1, \ldots, n-1\}$ such that $j \equiv x \pmod{n}$. We say that a diagonal of the given n-gon is *visible* if the entire n-gon is inside the angle with its vertex in T whose chord is that diagonal. For example, (observe image) diagonal $\overline{A_iA_k}$ $(i, k \in \mathbb{Z})$, with respect to X, is visible. (1 point)
$\overline{A_iA_k}$ is a visible diagonal. Inside the triangle A_iXA_k there are vertices $A_{i+1}, A_{i+2}, \ldots, A_{k-1}$ and outside of it all the remaining vertices of the given n-gon. So, the area $F(X)$ is equal to the sum of areas of the triangle A_iXA_k and polygon $A_kA_{k+1}\ldots A_{i-1}A_i$. (1 point)
Let l be a line through X parallel to the line A_iA_k and let A and B be the intersections of l and lines $A_{i-1}A_i$ i $A_{k+1}A_k$, respectively. (1 point)
If $F(X) = 1$, then X is an element of S. Then, it is obvious that all points T that are on line l and for which $\overline{A_iA_k}$ is a visible diagonal are also elements of set S. Furthermore, diagonal $\overline{A_iA_k}$ is no longer visible to the point T as T moves on l and becomes colinear with some side of the given n-gon. So, all points T of line l that are possibly in S (which they are if X is in S) are on the segment $\overline{AB}$. (2 points)
Now we can easily deduce the following conclusion: By observing all points $T \in S$ we can see that, as they "go around" the given n-gon, with them also rotate their respective visible diagonals. Every point of the given n-gon can, therefore, become one endpoint of some of the visible diagonals (visible diagonals interesting to us – those who are visible with respect to some point from the set S)

exactly once, and only exactly once stop being the endpoint. Because of that, there can be at most $2n$ interesting visible diagonals. It is clear that we will get a convex polygon – as point T rotates around the given n-gon, segments $\overline{AB}$ rotate with her and one always "connects" itself to the other one.

(5 points)

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Problem 4. Prove that for every natural number k, there exist infinitely many natural numbers n such that

$$\frac{n - d(n^r)}{r} \in \mathbb{Z}, \text{ for every } r \in \{1, 2, \ldots, k\}.$$

Here, $d(x)$ denotes the number of natural divisors of a natural number x, including 1 and x itself.

(Melkior Ornik)

Solution. Let us first prove the following lemma.

Lemma 1. $d(n^r) \equiv 1 \pmod{r}$.

Proof. Let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_i^{\alpha_i}$, where $p_1 < p_2 < \ldots < p_i$ are all prime divisors of the natural number n and i, α_1, α_2, $\ldots$, α_i are natural numbers, be the prime factorization of n. Then,

$$n^r = p_1^{r\alpha_1} \cdot p_2^{r\alpha_2} \cdot \ldots \cdot p_i^{r\alpha_i},$$

that is, $d(n^r) = (r\alpha_1 + 1) \cdot (r\alpha_2 + 1) \cdot \ldots \cdot (r\alpha_i + 1)$. (1 point)
Clearly, for every $j \in \{1, 2, \ldots, i\}$, the following holds: $r\alpha_j + 1 \equiv 1 \pmod{r}$, so

$$d(n^r) \equiv (r\alpha_1 + 1) \cdot (r\alpha_2 + 1) \cdot \ldots \cdot (r\alpha_i + 1) \equiv 1^i \equiv 1 \pmod{r}.$$

This has proven the lemma. (1 point)

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Now, our problem (because of **Lemma 1**) boils down to showing that for every natural number k there exists infinitely many natural numbers n such that

$$r \mid n - 1, \quad \forall r \in \{1, 2, \ldots, k\}.$$

Let $n = m \cdot k! + 1$, m being a natural number. (1 point)
Now it is obvious that $n \equiv m \cdot k! + 1 \equiv$ [because of $r \mid k!$] $\equiv 1 \pmod{r}$ for every $r \in \{1, 2, \ldots, k\}$. Because of that, for this choice of number n we have $r \mid n - 1$ for every $r \in \{1, 2, \ldots, k\}$. (1 point)
As m can be any natural number, for every natural number k there exists infinitely many natural numbers n such that $r \mid n - 1$ for every $r \in \{1, 2, \ldots, k\}$. This is, because of **Lemma 1**, equivalent to the problem claim. (6 points)

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