3^{RD} European Mathematical Cup

6^{th} December 2014–14^{th} December 2014

Senior Category

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Problems and Solutions

Problem 1. Prove that there are infinitely many positive integers which can't be expressed as $a^{d(a)} + b^{d(b)}$ where a and b are positive integers.

For positive integer a expression $d(a)$ denotes the number of positive divisors of a. *(Borna Vukorepa)*

Solution. We will show that $a^{d(a)}$ is a square of an integer for every positive integer a.

If a is a square of an integer, **any its power is also a square of an integer.**

If a is not a perfect square, number of it's positive divisors is even. We can prove this by pairing divisiors of a as d and $\frac{a}{d}$. A divisor d won't be paired with itself because that would imply $a = d^2$. This proves that $d(a)$ is even and hence $a^{d(a)}$ is a perfect square for every positive integer a.

The expression in the problem is hence a sum of two squares. Every number of the form $4t+3$ can't be written as a sum of two squares because 0 and 1 are the only quadratic residues modulo 4, so it is impossible for a sum of two squares to give remainder 3 modulo 4.

Problem 2. Jeck and Lisa are playing a game on an $m \times n$ board, with $m, n > 2$. Lisa starts by putting a knight onto the board. Then in turn Jeck and Lisa put a new piece onto the board according to the following rules:

1. Jeck puts a queen on an empty square that is two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from Lisa's last knight.
2. Lisa puts a knight on an empty square that is on the same, row, column or diagonal as Jeck's last queen.

The one who is unable to put a piece on the board loses the game. For which pairs (m, n) does Lisa have a winning strategy?

(Stijn Cambie)

Solution. We shall show that Lisa has a winning strategy if and only if m and n are both odd.

Lisa's winning strategy

Suppose the game is played on an $m \times n$ board with m and n both odd. Then Lisa puts her first knight in a corner and partitions the remaining squares of the board into 'dominoes'. In each turn Jeck has to put a queen in one of these dominoes and Lisa puts a knight on the other square of the domino. As the board is finite, Jeck can't keep finding new dominoes and so Lisa will win.

Jeck's winning strategy

Suppose the game is played on an $m \times n$ board with m or n even. We shall show that Jeck is able to partition the board into pairs of squares that are two squares horizontally and one square vertically, or alternatively one square horizontally and two squares vertically, away from each other. In each turn Lisa has to put a knight in one of these and Jeck puts a queen on the other square of the pair. As the board is finite, Lisa can't keep finding new pairs and so Jeck will win. Now we prove that Jeck can make the required partition.

Case 1. Suppose $4|m$ or $4|n$. We know that any $k \times 4l$ board ($k \geq 2$) can be divided into 2×4 and 3×4 boards (firstly divide $k \times 4l$ board in l boards of dimensions $k \times 4$; after that every $k \times 4$ board divide in $\frac{k}{2}$ boards of dimensions 2×4, or in $\frac{k-3}{2}$ boards of dimensions 2×4 and one 3×4 board, dependently on parity of k). The following diagrams show that every 2×4 and every 3×4 board allows a required partition.

1	2	3	4
3	4	1	2

1	2	3	4
3	5	1	6
2	6	4	5

1 point.

Case 2. Suppose $m, n \equiv 1, 2 \pmod 4$. Any $(5+4k) \times (6+4l)$ board can be divided into a 5×6 board, a $4k \times 6$ board, a $5 \times 4l$ board and a $4k \times 4l$ board. The following diagram shows that a 5×6 board allows a required partition.

1	2	14	13	12	11
3	4	12	11	14	15
2	1	13	15	7	8
4	3	5	6	9	10
5	6	9	10	8	7

According to case 1 a $4k \times 6$ board, a $5 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Case 3. Suppose $m, n \equiv 2, 3 \pmod 4$. Any $(3+4k) \times (6+4l)$ board can be divided into a 3×6 board, a $4k \times 6$ board, a $3 \times 4l$ board and a $4k \times 4l$ board. The following diagram shows that a 3×6 board allows a required partition.

1	2	3	4	7	8
3	4	1	6	9	5
2	6	9	5	8	7

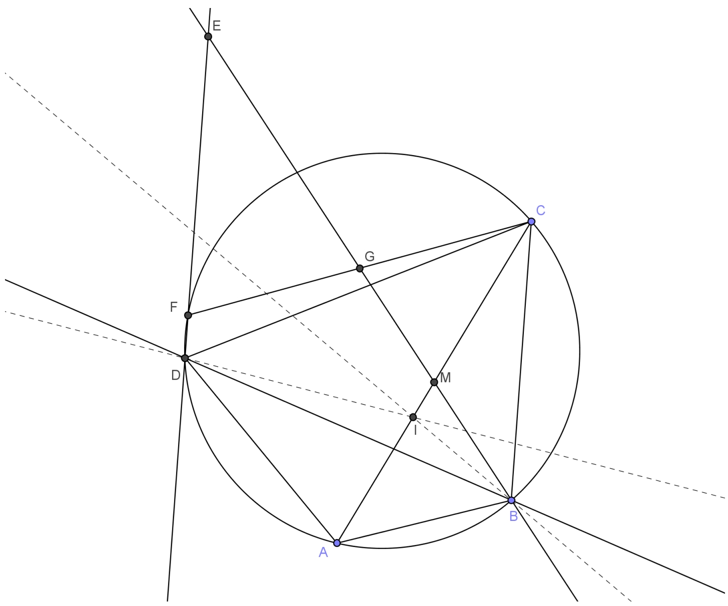
According to case 1 a $4k \times 6$ board, a $3 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Case 4. Suppose $m, n \equiv 2 \pmod 4$ .Any $(6+4k) \times (6+4l)$ board can be divided into a 6×6 board, a $4k \times 6$ board, a $6 \times 4l$ board and a $4k \times 4l$ board. The 6×6 board can be partitioned in two 3×6 boards, which were already solved. According to case 1 a $4k \times 6$ board, a $6 \times 4l$ board and a $4k \times 4l$ board also allow a partition.

Problem 3. Let $ABCD$ be a cyclic quadrilateral with the intersection of internal angle bisectors of $\angle ABC$ and $\angle ADC$ lying on the diagonal AC. Let M be the midpoint of AC. The line parallel to BC that passes through D intersects the line BM in E and the circumcircle of $ABCD$ at F where $F \neq D$. Prove that $BCEF$ is a parallelogram.

(Steve Dinh)

Solution. We prove the problem in reverse as this is much more natural in this problem.



We note that if $BCEF$ is a parallelogram then its diagonals are bisecting each other so the point $G \equiv BE \cap CF$ should be the midpoint of CF.

If G is the midpoint of CF then $\triangle GBC$ and $\triangle GEF$ are congruent as $CG = GF$ and $FE||BC$ gives $\angle GEF = \angle GBC$ and $\angle GFE = \angle GCB$. Hence this implies $BG = GE$ and in particular $BCEF$ is a paralelogram as its diagonals bisect each other. Hence G being midpoint of CF is equivalent to our problem.

As M is the midpoint of AC by the midline theorem applied to triangle ACF we have G is the midpoint of CG if and only if $MG||AF$. Hence we only need to prove $BM||AF$.

Now we further notice that, using $FD||BC$, this is equivalent to $\angle AFD = \angle MBC$.

We further see that $\angle AFD = \angle ABD$ as they are angles over the same chord. So our claim is equivalent to $\angle ABD = \angle MBC$.

We add that here depending on the relative position of F on the circles we might have $\pi - \angle AFD = \angle MBC$ but then $\pi - \angle AFD = \angle ABD$ so the final conclusion still holds.

We know that $\angle BDA = \angle BCM$ as they are angles over the same chord. Now this gives us that our claim is equivalent to the claim $\triangle BCM \sim \triangle BDA$.
The same angle equality shows that this is equivalent to $\frac{BC}{CM} = \frac{AD}{BD}$. Using the fact M is the midpoint of AC we have $CM = \frac{AC}{2}$ so our claim is equivalent to $2AD \cdot BC = BD \cdot AC$.
We further have by the angle bisector theorem applied to $\triangle ABC$ and $\triangle CDA$:

$$\frac{AB}{BC} = \frac{AI}{CI} = \frac{AD}{CD}$$

So using this our claim is equivalent to $AB \cdot CD + AD \cdot BC = BD \cdot AC$ which we can recognise to be the Ptolomeys theorem for cyclic quadrilaterals.

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ the following holds:

$$f\left(x^2\right) + f\left(2y^2\right) = (f(x+y) + f(y))(f(x-y) + f(y)).$$

(Matija Bucić)

Solution. Let $P(x, y)$ be the assertion $f\left(x^2\right) + f\left(2y^2\right) = \left(f(x+y) + f(y)\right)\left(f(x-y) + f(y)\right)$.
$P(0, x)$ gives us

$$f(0) + f\left(2x^2\right) = 2f(x)\left(f(x) + f(-x)\right) \tag{1}$$

and $P(0, -x)$ gives us

$$f(0) + f\left(2x^2\right) = 2f(-x)\left(f(x) + f(-x)\right). \tag{2}$$

By combining (1) and (2) we get

$$f(x)^2 = f(-x)^2. \tag{3}$$

$P(0, 0)$ gives us $2f(0) = 4f(0)^2$, thus we have two cases:

1. $f(0) = \frac{1}{2}$.
   $P(x, 0)$ gives us

$$f(x^2) = \left(f(x) + \frac{1}{2}\right)^2 - \frac{1}{2}, \tag{4}$$

   while $P(-x, 0)$ gives us

$$f(x^2) = \left(f(-x) + \frac{1}{2}\right)^2 - \frac{1}{2}. \tag{5}$$

   Combining (4) and (5) and using (3) we get

$$f(x) = f(-x). \tag{6}$$

   The assertion $P\left(x^2, x^2\right)$ can be written as

$$f\left(x^4\right) + f\left(2x^4\right) = \left(f\left(2x^2\right) + f\left(x^2\right)\right)\left(\frac{1}{2} + f\left(x^2\right)\right). \tag{7}$$

   For an arbitrary $x \in \mathbb{R}$, let us denote $a = f(x)$. Using (4) we get:

$$f\left(x^2\right) = \left(a + \frac{1}{2}\right)^2 - \frac{1}{2},$$
$$f\left(x^4\right) = \left(f\left(x^2\right) + \frac{1}{2}\right)^2 - \frac{1}{2} = \left(a + \frac{1}{2}\right)^4 - \frac{1}{2}.$$

   Using (1) and (6) we get:

$$f\left(2x^2\right) = 4f(x)^2 - \frac{1}{2} = 4a^2 - \frac{1}{2},$$
$$f(2x^4) = 4f(x^2)^2 - \frac{1}{2} = 4\left(\left(a + \frac{1}{2}\right)^2 - \frac{1}{2}\right)^2 - \frac{1}{2}.$$

   Plugging the last 4 equations in (7) we get:

$$\left(a + \frac{1}{2}\right)^4 + 4\left(\left(a + \frac{1}{2}\right)^2 - \frac{1}{2}\right)^2 - 1 = \left(4a^2 - 1 + \left(a + \frac{1}{2}\right)^2\right)\left(a + \frac{1}{2}\right)^2,$$

   which is equivalent to

$$\left(a + \frac{1}{2}\right)^2 (4a - 2) = 0.$$

Therefore $a = \pm\frac{1}{2}$ and $f(x) = \pm\frac{1}{2}$. Now if we use (6) in (1) we get

$$f(0) + f\left(2x^2\right) = 4(f(x))^2 = 1$$

so $f\left(2x^2\right) = \frac{1}{2}$ for every x, now using (6) we conclude $f(x) = \frac{1}{2}$ for all x which is easily checked to be a solution.

2. $f(0) = 0$.
We immediately see using $P(x, 0)$ that

$$f(x^2) = f(x)^2. \tag{8}$$

By comparing $P(x, y)$ and $P(x, -y)$ and using (3) we get:

$$(f(y) - f(-y))\,(f(x+y) + f(x-y)) = 0.$$

If there exists $c \in \mathbb{R}$ such that $f(c) \neq f(-c)$ we have for all x

$$f(x+c) = -f(x-c),$$

Plugging in $x + c$ in x here gives us:

$$f(x+2c) = -f(x). \tag{9}$$

Specially, $f(2c) = 0$. Now, $P(2c - y, y)$:

$$\begin{aligned} f((2c-y)^2) + f(2y^2) &= (f(2c) + f(y))(f(2c-2y) + f(y)), \\ (-f(-y))^2 + f(2y^2) &= f(y)f(2c-2y) + f(y)^2, \\ f(2y^2) &= f(y)f(2c-2y) = -f(y)f(-2y) \end{aligned} \tag{10}$$

Let $S(x)$ denote the statement $(x \neq 0) \wedge (f(x) = f(-x) \neq 0)$. If there is no $d \in \mathbb{R}$ such that $S(d)$ then $f(x) = -f(-x)$ for all $x \in \mathbb{R}$. $P(0, x)$ gives us

$$f(2x^2) = 2f(x)(f(x) + f(-x)) = 0,$$

which gives us another solution $f(x) = 0$. Now, let us assume that there exists $d \in \mathbb{R}$ such that $S(d)$ holds. Obviously, $S(-d)$ holds, as well. $P(0, d)$ gives us

$$f(2d^2) = 4f(d)^2$$

and (10) gives us

$$\begin{aligned} f(2d^2) &= -f(d)f(-2d) \\ f(-2d) &= -4f(d) \\ f(2d) &= -4f(-d) = -4f(d) = f(-2d) \end{aligned}$$

Therefore, $S(2d)$ also holds. Inductively, we deduce that $S(2^n d)$ holds for every $n \in \mathbb{N}$. Also, $f(2^n d) = (-4)^n f(d)$, which means that f is unbounded.
$P(x, c)$, using the fact $f(x^2) = f(x)^2$:

$$f(x)^2 + f(2c^2) = f(x+c)f(x-c) + f(c)(f(x+c) + f(x-c)) + f(c)^2,$$

and since $f(x+c) = -f(x-c)$ and $f(2c^2) = 0$ (this follows from $P(0, c)$) we have

$$f(x)^2 + f(x+c)^2 = f(c)^2,$$

which implies that f is bounded and that is contradiction. Therefore, there is no $c \in \mathbb{R}$ such that $f(c) = -f(c)$ and therefore

$$f(x) = f(-x), \quad \text{for all } x \in \mathbb{R}. \tag{11}$$

$P(0, x)$:

$$f(2x^2) = 4f(x)^2 = 4f(x^2).$$

Therefore, using (11)

$$f(2x) = 4f(x), \quad \text{for all } x \in \mathbb{R}. \tag{12}$$

$P(x, y)$ can now be written as follows:

$$f(x)^2 + 3f(y)^2 = f(y)(f(x+y) + f(x-y)) + f(x+y)f(x-y),$$

and similarly, $P(y, x)$ can be written as

$$f(y)^2 + 3f(x)^2 = f(x)(f(x+y) + f(x-y)) + f(x+y)f(x-y).$$

subtracting the previous two equalities

$$(f(x) - f(y))(2f(x) + 2f(y) - f(x+y) - f(x-y)) = 0. \tag{13}$$

Assume that for some $x, y \in \mathbb{R}$ $f(x) = f(y) = a$. Let $f(x+y) = b$ and $f(x-y) = c$.

Now we have:

$$4a^2 = bc + ab + ac \tag{14}$$

$P(x+y, x-y)$:

$$f(x+y)^2 + 4f(x-y)^2 = (f(2x) + f(x-y))(f(2y) + f(x-y)),$$

i.e.

$$b^2 + 4c^2 = (4a + c)^2 \tag{15}$$

If we plug in $x \to x+y$, $y \to x-y$ in (13) we get

$$(f(x+y) - f(x-y))(2f(x+y) + 2f(x-y) - f(2x) - f(2y)) = 0$$

i.e.

$$(b-c)(2b + 2c - 8a) = 0.$$

If $b = c$ (15) gives us

$$5b^2 = (4a+b)^2,$$
$$b^2 = 4a^2 + 2ab$$

while (14) gives us

$$4a^2 = b^2 + 2ab$$

Thus, $ab = 0$ and $a = b = c = 0$ which implies $2a + 2a - b - c = 0$. On the other hand, if $b \neq c$ we also have $2a + 2a - b - c = 0$

Therefore, $f(x) = f(y)$ implies $2f(x) + 2f(y) = f(x+y) + f(x-y)$ while $f(x) \neq f(y)$, using (13) also implies $2f(x) + 2f(y) = f(x+y) + f(x-y)$.

Therefore, for all x, y:

$$2f(x) + 2f(y) = f(x+y) + f(x-y) \tag{16}$$

Now we have:

$$\begin{aligned} f(x)^2 + 3f(y)^2 &= f(x+y)f(x-y) + f(y)(f(x+y) + f(x-y)) \\ &= f(x+y)f(x-y) + f(y)(2f(x) + 2f(y)), \\ (f(x) - f(y))^2 &= f(x+y)f(x-y). \end{aligned} \tag{17}$$

Combining (16) i (17) gives us

$$(f(x+y) - f(x) - f(y))^2 = 4f(x)f(y). \tag{18}$$

Let $g : \mathbb{R} \to \mathbb{R}_0^+$ be the function such that $f(x) = g(x)^2$. Equations (8), (11) and (12) imply that $g(x^2) = g(x)^2$, $g(-x) = g(x)$ and $g(2x) = 2g(x)$, respectively.

Equation (16) can be written as

$$f(x+y) - f(x) - f(y) = -(f(x-y) - f(x) - f(y)).$$

If $f(x+y) - f(x) - f(y) \geqslant 0$, from (18) we conclude that $g(x+y) = g(x) + g(y)$. Otherwise, $f(x-y) - f(x) - f(y) \geqslant 0$ and equation (18) can be rewritten as

$$(f(x-y) - f(x) - f(y))^2 = 4f(x)f(y).$$

From the last equation we can conclude that $g(x-y) = g(x) + g(y)$.

Therefore

$$g(x+y) = g(x) + g(y) \quad \text{or} \quad g(x-y) = g(x) + g(y) \tag{19}$$

and thus one of the following two equations hold:

$$g(x^2 + y^2) + 2g(xy) = g(x^2 + y^2 + 2xy) = g(x+y)^2 \tag{20}$$

or

$$g(x^2 + y^2) + 2g(xy) = g(x^2 + y^2 - 2xy) = g(x-y)^2 \tag{21}$$

From (18) we conclude:

$$g(x+y) = g(x) + g(y) \quad \text{or} \quad g(x+y) = |g(x) - g(y)|. \tag{22}$$

By putting $-y$ instead of y in (22) and using $g(-y) = g(y)$ we get:

$$g(x-y) = g(x) + g(y) \quad \text{or} \quad g(x-y) = |g(x) - g(y)|. \tag{23}$$

Equations (22) and (23) imply that each of $g(x-y)^2$ and $g(x+y)^2$ can be written as either $(g(x) + g(y))^2$ or $(g(x) - g(y))^2$. Thus, no matter whether (20) or (21) holds, one of the following equations must hold:

$$g(x^2 + y^2) + 2g(xy) = (g(x) + g(y))^2 \tag{24}$$

or

$$g(x^2+y^2)+2g(xy)=(g(x)-g(y))^2 \tag{25}$$

Without loss of generality we may assume that $g(x) \geqslant g(y)$. If $g(x^2+y^2)=g(x)^2+g(y)^2$ then equations (24) i (25) imply that $g(xy)=g(x)g(y)$ or $g(xy)=-g(x)g(y)$ and because g is non-negative we conclude that $g(xy)=g(x)g(y)$. Otherwise, $g(x^2+y^2)=|g(x^2)-g(y^2)|=g(x)^2-g(y)^2$ and we have

$$g(x)^2+g(y)^2 \pm 2g(x)g(y)=g(x)^2-g(y)^2+2g(xy)$$

and

$$g(y)^2 \pm g(x)g(y)=g(xy).$$

However, since $g(x) \geqslant g(y)$ and $g(xy) \geqslant 0$ we get

$$g(y)^2+g(x)g(y)=g(xy).$$

Therefore, we conclude that

$$g(xy)=g(y)^2+g(x)g(y) \quad (\text{for } g(y) \leqslant g(x)) \quad \text{or} \quad g(xy)=g(x)g(y). \tag{26}$$

Thus,

$$g(xy) \geqslant g(x)g(y). \tag{27}$$

If for some a, b it holds that $g(a^2+b^2) \neq g(a)^2+g(b)^2$ we may assume that $g(a)>g(b)$ and we have $g(a^2+b^2)=g(a)^2-g(b)^2$, and

$$g(ab)=g(b)^2+g(a)g(b).$$

Let us denote $a'=2a$ and $b'=\frac{1}{2}b$. We have $g(a')=2g(a)$ and $g(b')=\frac{1}{2}g(b)$. Therefore, $g(a')>g(a)>g(b)>g(b')$. Note that $g(a'b')=g(ab)$ and $g(a')g(b')=g(a)g(b)$. From (26) we conclude that either $g(a'b')=g(a')g(b')$ or $g(a'b')=g(b')^2+g(a')g(b')$. Each of these two cases is only possible when $g(b)=0$. However, this implies that $g(a^2+b^2)=g(a^2)-g(b^2)=g(a^2)+g(b^2)$ which is a contradiction.

Therefore, there are no a, b such that $g(a^2+b^2) \neq g(a)^2+g(b)^2$ and for all $x, y \geqslant 0$ $g(x+y)=g(x)+g(y)$ which, together with the fact that g is non-negative, means that g satisfies a Cauchy functional equation whose only solution is $g(x)=g(1)x$. Since $g(1)=g(1)^2$ we get that $g(1)=1$ and $f(x)=x^2$ for all x.

Therefore there are 3 solutions which are given by

- $f(x)=0 \;\; \forall x \in \mathbb{R}$,
- $f(x)=\frac{1}{2} \;\; \forall x \in \mathbb{R}$ and
- $f(x)=x^2 \;\; \forall x \in \mathbb{R}$.