3^{RD} European Mathematical Cup

6^{th} December 2014–14^{th} December 2014

Junior Category

MLADI NADARENI MATEMATIČARI
Marin Getaldić

Problem 1. Which of the following claims are true, and which of them are false? *If a fact is true you should prove it, if it isn't, find a counterexample.*

a) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

b) Let a, b, c be real numbers such that $a^{2014} + b^{2014} + c^{2014} = 0$. Then $a^{2015} + b^{2015} + c^{2015} = 0$.

c) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$ and $a^{2015} + b^{2015} + c^{2015} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

(Matko Ljulj)

Problem 2. In each vertex of a regular n-gon $A_1A_2...A_n$ there is a unique pawn. In each step it is allowed:

1. to move all pawns one step in the clockwise direction or
2. to swap the pawns at vertices A_1 and A_2.

Prove that by a finite series of such steps it is possible to swap the pawns at vertices:

a) A_i and A_{i+1} for any $1 \leqslant i < n$ while leaving all other pawns in their initial place

b) A_i and A_j for any $1 \leqslant i < j \leqslant n$ leaving all other pawns in their initial place.

(Matija Bucić)

Problem 3. Let ABC be a triangle. The external and internal angle bisectors of $\angle CAB$ intersect side BC at D and E, respectively. Let F be a point on the segment BC. The circumcircle of triangle ADF intersects AB and AC at I and J, respectively. Let N be the mid-point of IJ and H the foot of E on DN. Prove that E is the incenter of triangle AHF.

(Steve Dinh)

Problem 4. Find all infinite sequences $a_1, a_2, a_3, \ldots$ of positive integers such that

a) $a_{nm} = a_n a_m$, for all positive integers n, m, and

b) there are infinitely many positive integers n such that $\{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}$.

(Matko Ljulj)

Time allowed: 240 minutes.

Each problem is worth 10 points.

Calculators are not allowed.