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Junior Category

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Problems and Solutions

Problem 1. Which of the following claims are true, and which of them are false? *If a fact is true you should prove it, if it isn't, find a counterexample.*

a) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

b) Let a, b, c be real numbers such that $a^{2014} + b^{2014} + c^{2014} = 0$. Then $a^{2015} + b^{2015} + c^{2015} = 0$.

c) Let a, b, c be real numbers such that $a^{2013} + b^{2013} + c^{2013} = 0$ and $a^{2015} + b^{2015} + c^{2015} = 0$. Then $a^{2014} + b^{2014} + c^{2014} = 0$.

(Matko Ljulj)

Solution. Firstly, we know that for every real number x, $x^2 \geqslant 0$ holds.
The key idea in this problem is to realize that the expression $a^{2014} + b^{2014} + c^{2014}$ is a sum of squares (which are nonnegative numbers). Thus $a^{2014} + b^{2014} + c^{2014} = 0 \iff a = b = c = 0$.

a) NO: It is sufficient to find three real numbers whose sum equals 0, and then take their 2013^{th} roots. For example: $a = \sqrt[2013]{1}$, $b = \sqrt[2013]{2}$, $c = \sqrt[2013]{-3}$.

b) YES: From the key idea we conclude $a = b = c = 0$, and then we conclude $a^{2015} + b^{2015} + c^{2015} = 0 + 0 + 0 = 0$.

c) NO: Again we have to find a counterexample, for instance $a = 1, b = 0, c = -1$.

Problem 2. In each vertex of a regular n-gon $A_1A_2...A_n$ there is a unique pawn. In each step it is allowed:

1. to move all pawns one step in the clockwise direction or
2. to swap the pawns at vertices A_1 and A_2.

Prove that by a finite series of such steps it is possible to swap the pawns at vertices:

a) A_i and A_{i+1} for any $1 \leqslant i < n$ while leaving all other pawns in their initial place

b) A_i and A_j for any $1 \leqslant i < j \leqslant n$ leaving all other pawns in their initial place.

(Matija Bucić)

Solution. We denote a pawn that was initially at point A_i as i. We will prove part a) and then use it to show part b).

a) We apply first operation $i - 1$ times which will bring i and $i + 1$ to points A_1 and A_2 and move every other pawn $i - 1$ steps in clockwise direction.
We can now apply second operation to swap i and $i + 1$ as they are at points A_1 and A_2. This does not affect the position of any other pawn.
We now apply first operation $n - i + 1$ times returning pawn $k \neq i, i + 1$ to point A_k while moving pawn i to A_{i+1} and pawn $i + 1$ to A_i which is exactly what we wanted.

b) We present 2 possible solutions, one using induction and one not using induction.
Solution 1: By using the previous problem we can swap pawns $(i, i + 1)$ as they are at points (A_i, A_{i+1}) then $(i, i+2)$ as they are at points (A_{i+1}, A_{i+2}) and carry on until we swap (i, j) as they were at points (A_{j-1}, A_j). This brings us to the state where i is at A_j and each $i + 1 \leqslant k \leqslant j$ is at point A_{k-1}.
We can now apply part a to swap j with $j - 1$ and similarly carry on till we swap j with $i + 1$. This will place j at A_i and move each $i + 1 \leqslant k \leqslant j - 1$ to A_k.
This brings us to the state where we swapped pawns i and j leaving others where they were just as was desired. □

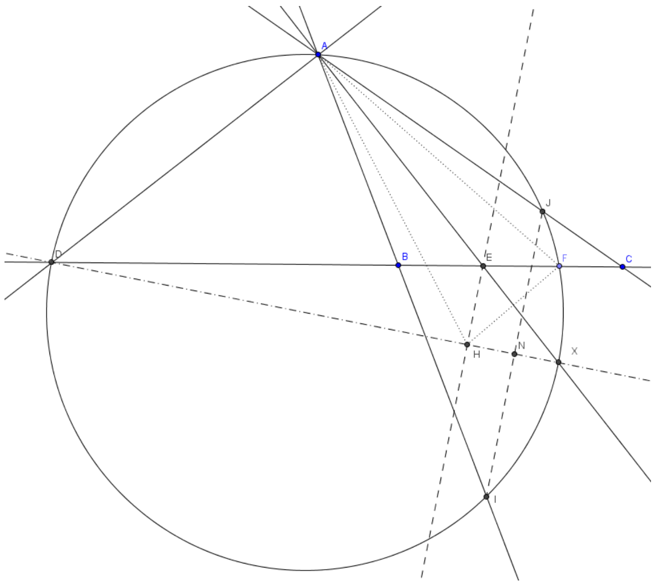
Solution 2**:** We use induction on n for the following claim:
We can swap any two pawns $1 \leqslant i < j \leqslant k$.
We note that the basis is exactly part a.
We assume we the claim holds for some k.
Hence we can swap any pawns $1 \leqslant i < j \leqslant k$ and only need to show that we can swap i and $k+1$ for any $1 \leqslant i \leqslant k$.
This follows as we can swap i and k then k and $k+1$ by part a). then again $k+1$ and i as they are now on points A_k and A_i.

□

Problem 3. Let ABC be a triangle. The external and internal angle bisectors of $\angle CAB$ intersect side BC at D and E, respectively. Let F be a point on the segment BC. The circumcircle of triangle ADF intersects AB and AC at I and J, respectively. Let N be the mid-point of IJ and H the foot of E on DN. Prove that E is the incenter of triangle AHF.

(Steve Dinh)

Solution. Denote by ω the circumcircle of $\triangle AHF$.



The key idea in the problem is to introduce a new point X which we define as the second intersection of DN and ω.
We now note that the $\angle JAD = \angle CAD = 90^\circ \pm \frac{\alpha}{2}$ and $\angle IAD = \angle BAD = 90^\circ \pm \frac{\alpha}{2}$ where $\alpha = \angle CAB$. As AD is an external bisector of $\angle CAB$.
The $\pm$ signs depend on the picture and student shouldn't be deducted any points for not noticing this.
Hence we have either $\angle JAD = \angle BAD$ or $\angle JAD + \angle IAD = 180^\circ$ so in both cases $DI = DJ$.
Now as N is midpoint of IJ this means that DN is bisector of IJ and hence pasess through the centre of the. This shows that DX is a diameter of ω and $EH||IJ$.
We also notice that $\angle EAD = 90^\circ$ as angle between bisectors and $\angle XAD = 90^\circ$ as DX is a diameter. Hence X, A, E are collinear.
Now this gives us $\angle DHE = \angle XHE = 90^\circ$ and $\angle XFE = \angle DFE = 90^\circ$ as DX is a diameter of ω and finally again $\angle EAD = 90^\circ$. All this gives us that quadrilaterals $XFEH$ and $ADEH$ are cyclic.
Final step is to use some angle chasing to get $\angle AHE = \angle ADH = \angle AXF = \angle EXF = \angle EHF$ where first, second and fourth equalities are due to cyclicity of $ADEH$, $ADXF$ and $XFEH$ respectively. Also $\angle DFH = \angle EFH = \angle EXH = \angle AFD = \angle AFE$ where the second and forth equalities are due to cyclicity of $XFEH$ and $ADXF$ respectively. This shows E is the incenter of $\triangle AFH$ as desired.

Problem 4. Find all infinite sequences $a_1, a_2, a_3, \ldots$ of positive integers such that

a) $a_{nm} = a_n a_m$, for all positive integers n, m, and
b) there are infinitely many positive integers n such that $\{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}$.

(Matko Ljulj)

Solution. Instead of sequence a_n, we'll use notation with the function $f(n)$ with same properties.

There exists only one such function: $f(n) = n$. We'll solve the problem with many separate facts.

Fact 1: $f(1) = 1$.

Proof: According to a) it holds $f(1) = f(1)f(1) = f(1)^2$. Since $f(1)$ is positive integer, it can't be $f(1) = 0$, so it must be $f(1) = 1$.

Fact 2: Function f is bijective.

Proof: Firstly, we'll show that f is injective. Let $a \neq b$ be two arbitrary positive integers and let's assume $f(a) = f(b)$. Since $\{1, 2, \ldots, n\} = \{f(1), f(2), \ldots, f(n)\}$ holds for infinitely many positive integers n, it holds for some integer greater than a and b. Then, since $f(a) = f(b)$, set $\{f(1), f(2), \ldots, f(n)\}$ contains $n-1$ or less (different) elements, but according to b), it contains n elements.

Secondly, we'll show that f is surjective. Let c be arbitrary integer and let's assume that $f(n) \neq c$ for all positive integers n. Similarly as in first part of proof, let's take positive integer n such that $\{1, 2, \ldots, n\} = \{f(1), f(2), \ldots, f(n)\}$ holds. Since $c \in \{1, 2, \ldots, n\}$, c is also element of the set $\{f(1), f(2), \ldots, f(n)\}$, so there exists positive integer $m \leqslant n$ such that $f(m) = c$.

Fact 3: Positive integer n is prime if and only if $f(n)$ is prime.

Proof: Let's assume that n is prime, but $f(n)$ isn't. Then it must be $f(n) = a'b' = f(a)f(b) = f(ab)$, where a', b' are positive integers greater than 1, and a, b are unique positive integers such that $f(a) = a', f(b) = b'$ (they exist since f is bijective). Since f is injective, $f(1) = 1$ and a', b' are not equal to 1, integers a, b are also not equal to 1. Since f is injective and $f(n) = f(ab)$, we have $n = ab$, so n is composite.

Let's assume that $f(n)$ is prime, but n isn't. Then there exist positive integers a, b greater than one such that $n = ab$. From there we have $f(n) = f(ab) = f(a)f(b)$. Again from injectivity of f and $f(1) = 1$, we see that $f(n)$ is product of two integers greater than 1.

Fact 4: If $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ is unique factorization of positive integer n, then

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \ldots f(p_k)^{a_k}$$

is unique factorization of positive integer $f(n)$.

Proof: From multiple use of the condition a) we get identity $f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \ldots f(p_k)^{a_k}$. From Fact 3, numbers $f(p_i)$ are prime. Since f is injective, none of two numbers $f(p_i)$ and $f(p_j)$ are equal.

Fact 5: (Technical result) For all positive integers $y < x$ there exist positive integer n_0 such that for all positive integers $n \geqslant n_0$ holds inequality

$$y^{n+1} < x^n.$$

Proof: It is sufficient to prove the fact only for consecutive integers y and $y+1$ (because we'll have $y^{n+1} < (y+1)^n \leqslant x^n$). By binomial theorem we have

$$(y+1)^n \geqslant y^n + ny^{n-1} = y^{n-1}(y+n).$$

Thus if we define $n_0 = y^2 - y + 1$, then for all $n \geqslant n_0$ we have

$$(y+1)^n \geqslant y^{n-1}(y+n) \geqslant y^{n-1}(y+n_0) = y^{n-1}(y^2+1) > y^{n+1}.$$

Another proof: Inequality is equivalent to

$$\left(\frac{x}{y}\right)^n > y.$$

The fact follows from the fact that the expression on the left hand side is increasing and it is unbounded, while the right hand side is fixed.

Fact 6: For all prime numbers p we have $f(p) \leqslant p$.

Proof: Let $p_1, p_2, \ldots, p_n, \ldots$ be the increasing sequence $2, 3, 5, 7, \ldots$ of all prime numbers. Let's take arbitrary prime number p_n. From the Fact 3 we have that $f(p_n)$ is also a prime. Let's take positive integer n_0 as the integer from the Fact 5, for positive integers $y = p_n < p_{n+1} = x$. Since b) holds for infinitely many positive integers, it holds for some positive integer N such that $\{1, 2, \ldots, N\} = \{f(1), f(2), \ldots, f(N)\}$, and such that $N \geqslant p_n^{n_0}$. Let α be the greatest positive integer such that $p_n^\alpha \leqslant N$. From definitions of N and α we have $\alpha \geqslant n_0$.

In set $\{1, 2, \ldots, N\}$ we'll observe all positive integers which are α^{th} power of a prime number. Since $N \geqslant p_n^\alpha$, we have that p_n^α is in that set. It is easy to see that all numbers $p_1^\alpha, \ldots, p_{n-1}^\alpha$ are also in that set. On the contrary, number p_{n+1}^α is not in that set, because from the definition of α and N respectively we have $N < p_n^{\alpha+1} \leqslant p_{n+1}^\alpha$ (remember Fact 5 and $\alpha \geqslant n_0$). Similarly, neither of the numbers p_m^α (for $m > n$) is not in the set $\{1, 2, \ldots, N\}$.

Let us now observe all positive integers which are α^{th} power of a prime and they are in the set $\{f(1), f(2), \ldots, f(N)\}$. According to Fact 4, we have that $f(n)$ is α^{th} power of a prime if and only if n is α^{th} power of a prime. From that and from previous paragraph we conclude that only such numbers are $f(p_1^\alpha), \ldots, f(p_n^\alpha)$.

Now we have $\{p_1^\alpha, \ldots, p_n^\alpha\} = \{f(p_1^\alpha), \ldots, f(p_n^\alpha)\}$. Thus $f(p_n^\alpha) \in \{p_1^\alpha, \ldots, p_n^\alpha\}$, so $f(p_n^\alpha) = p_i^\alpha$ for some $1 \leqslant i \leqslant n$, which implies $f(p_n)^\alpha = p_i^\alpha$, for some $1 \leqslant i \leqslant n \implies f(p_n) = p_i \leqslant p_n$, which completes the proof.

Fact 7: For every positive integer we have $f(n) = n$.

Proof: From Fact 3 we have that $f(p)$ if and only if p is prime. Let $p_1, p_2, \dots, p_n, \dots$ be the increasing sequence $2, 3, 5, 7, \dots$ of all prime numbers. From Fact 6 we have $f(p_1) \leqslant p_1 \implies f(2) = 2$. For $n \geqslant 2$, inductively and from injectivity of f we have $f(p_n) > p_{n-1}$ and from Fact 6 we have $f(p_n) \leqslant p_n$, thus is must be $f(p_n) = p_n$, for all positive integers n.

Now for arbitrary positive integer n from Fact 4 we have

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \dots f(p_k)^{a_k} = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k} = n,$$

which completes our proof.

Remark: We can prove Fact 6 differently (without using Fact 5). We observe numbers $1 \cdot 2 \cdot \ldots \cdot n$ and $f(1) \cdot f(2) \cdot \ldots \cdot f(n)$, and their unique factorizations. They coincide for infinitely many positive integers n. For fixed primes p, q, if we take sufficiently great n, we can use well-known formula for $\nu_p(n!)$ to prove that $\nu_p(n!) > \nu_q(n!)$ for all $q > p$ (here positive integer n depends on p, q).